Recursive construction

Harmonic numbers

$$H_1 = 1$$
$$H_n = H_{n-1} + \frac{1}{n} \quad \text{for all } n \geq 2$$

Fibonacci numbers

$$F_0 = 0$$
$$F_1 = 1$$
$$F_n = F_{n-1} + F_{n-2} \quad \text{for all } n \geq 2$$

Sequence

$$a_0 = 0$$
$$a_1 = 1$$
$$a_2 = 2$$
$$a_n = a_{n-1} + a_{n-2} + a_{n-3} \quad \text{for all } n \geq 3$$

Matrix multiplication

$$M_0 M_1 M_2 \cdots M_n$$

$$M_0 M_1 M_2 M_3$$

$$((M_0 M_1) M_2) M_3$$

$$(M_0 (M_1 M_2)) M_3$$

$$(M_0 M_1)(M_2 M_3)$$

$$M_0 ((M_1 M_2) M_3)$$

$$M_0 (M_1 (M_2 M_3))$$

Associativity of matrix multiplication

$$(AB)C = A(BC)$$

Holds for any associative operation

$\wedge, \vee$ propositions, $\cup, \cap$ of sets

Recursive construction

$n=0 \quad M_0$

$n=1 \quad M_0 M_1$

For $n \geq 2$, define

$$M_0 M_1 M_2 \cdots M_n = (M_0 M_1 M_2 \cdots M_{n-1}) M_n$$

$$(\cdots(((M_0 M_1) M_2) M_3) \cdots M_{n-1}) M_n$$

Claim: $M_0 M_1 M_2 \cdots M_n$ is independent of parenthesization.

Proof: [Strong form of induction] $n \geq 2$

Basis $\quad M_0 M_1 M_2 = (M_0 M_1) M_2 = M_0 (M_1 M_2)$

law of associativity

Inductive step $\quad M_0 M_1 \ldots M_r$ are indepent of parenthesization for $r = 0, 1, 2, \ldots, n-1$

$$M_0 M_1 M_2 \cdots M_n = (M_0 M_1 \cdots M_{n-1}) M_n$$

↪ take any parethesization of this.

$$(M_0 M_1 \cdots M_r)(M_{r+1} \cdots M_n)$$

$r = n-1$

$$\underbrace{(M_0 M_1 \cdots M_{n-1})}_{\text{independent of parethesization}} M_n$$ — this is the interpretation

$r < n-1$

$$(M_0 M_1 \cdots M_r)(M_{r+1} \cdots M_{n-1} M_n)$$

by induction is independent of parenthesization

$$= (M_0 M_1 \cdots M_r)((M_{r+1} \cdots M_{n-1}) M_n)$$

[by our interpretation]

$$= [(M_0 M_1 \cdots M_r)(M_{r+1} \cdots M_{n-1})] M_n$$

$$= (M_0 M_1 \cdots M_{n-1}) M_n$$

$M_0 M_1 M_2 \cdots M_n$

The number of parenthesization of $M_0 M_1 M_2 \cdots M_n$ is equal to the n-th Catalan number $C(n)$.

$A = \{$ valid paths from $(0,0)$ to $(n,n)\}$

$B = \{$ parenthesization of $M_0 M_1 M_2 \cdots M_n\}$

To show that these two sets are of the same size

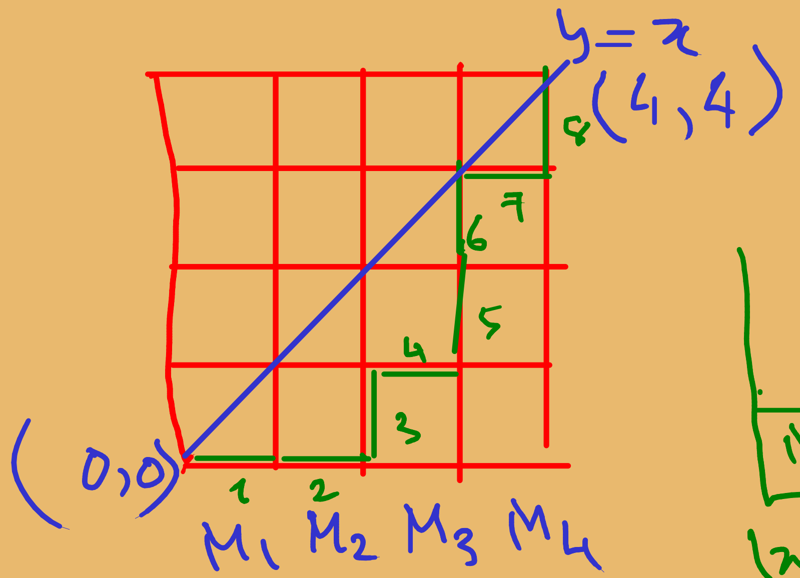
$$f : A \to B \qquad g : B \to A$$
$$P \mapsto Q \qquad Q \mapsto P$$

$$g \circ f = id_A \;/\; f \circ g = id_B$$

The contruction

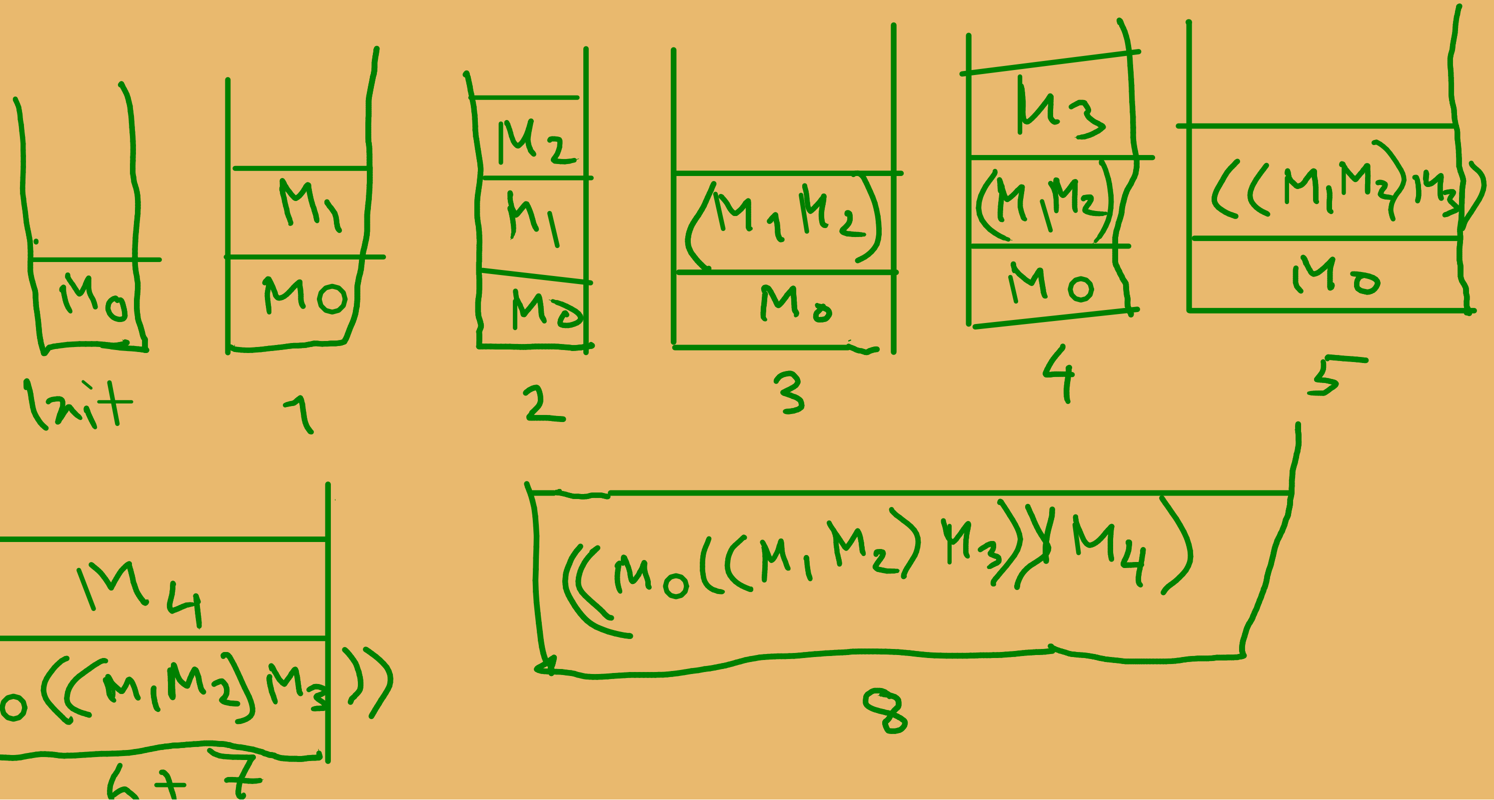
$$f : \{\text{paths}\} \to \{\text{parenthesizations}\}$$

$n = 4$



$M_0 M_1 M_2 M_3 M_4$

Stack



Define the Catalan numbers $C(n)$ recursively

$C(n) = \#$ of parenthesizations of $M_0 M_1 M_2 \cdots M_n$

$$C(n) = \frac{1}{n+1}\binom{2n}{n}$$

$$C(0) = 1$$

$n \geq 1$ $\quad (M_0 M_1 \cdots M_r)(M_{r+1} \cdots M_n)$

$0 \leq r \leq n-1$

$$C(n) = \sum_{r=0}^{n-1} C(r)\,C(n-r-1)$$

$$= C(0)C(n-1) + C(1)C(n-2) + C(2)C(n-3) + \cdots + C(n-1)C(0)$$

By strong induction, $C(n)$ is defined for all $n \geq 0$.

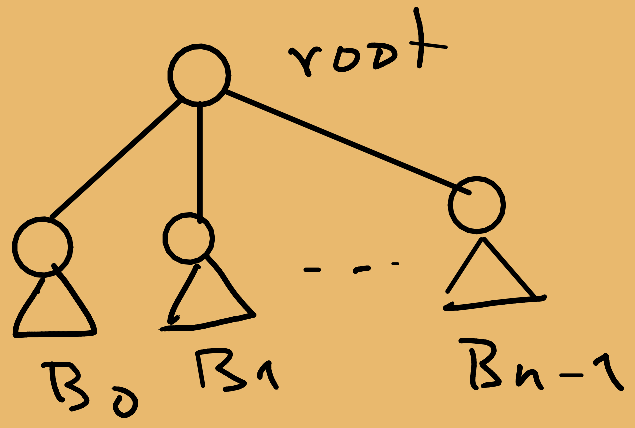
Binomial Trees

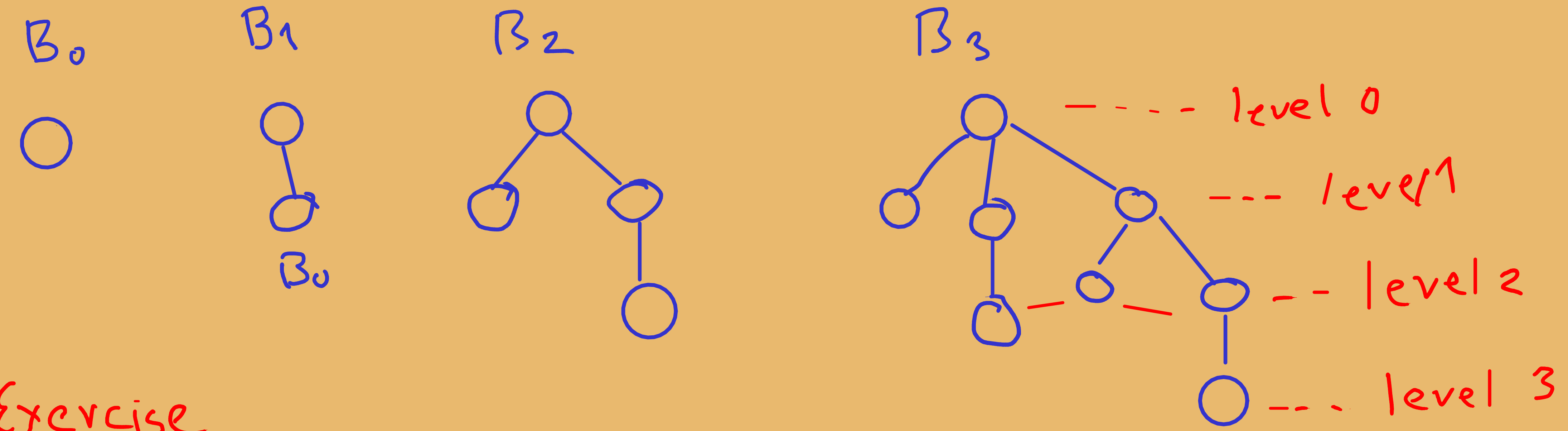
B_n, $n \geq 0$.

Basis $(n = 0)$ a single-node tree ○ root

Induction $B_0, B_1, \ldots, B_{n-1}$

Construct B_n as follows:





largest level in the tree
height of the tree

Exercise

Prove the following about B_n.

(1) B_n contains 2^n nodes $(n \geq 0)$

(2) B_n contains 2^{n-1} leaf nodes $(n \geq 1)$

(3) The height of B_n is n.

(4) There are $\binom{n}{i}$ nodes at level i for all $i = 0, 1, 2, \ldots, n$.