Proof by induction

$\mathbb{N} = \mathbb{Z}^+ = \mathbb{Z}_{>0} = \{ n \in \mathbb{Z} \mid n > 0 \}$

$\mathbb{Q}^+$ = set of all +ve rational numbers

$\mathbb{R}^+$ = set of all +ve real numbers

$\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$

Well-ordering principle

Let $S \subseteq \mathbb{Z}^+$, $S \neq \emptyset$. Then S contains a smallest element.

$\mathbb{Z}^+$ is well-ordered.

$\{ \frac{1}{n} \mid n \in \mathbb{Z}^+ \}$ $\quad \mathbb{Q}^+, \mathbb{R}^+$ are not well-ordered.

Principle of mathematical induction (Weak form)

Let $S(n)$ be a statement about $n \in \mathbb{Z}^+$ such that the following two conditions hold.

(1) $S(1)$ is true.

(2) $S(n)$ is true $\Rightarrow$ $S(n+1)$ is true for all $n \in \mathbb{Z}^+$

Then $S(n)$ is true for all $n \in \mathbb{Z}^+$.

Proof: $A \subseteq \mathbb{Z}^+$ s.t. $S(n)$ is not true whenever $n \in A$. Assume $A \neq \emptyset$. Pick $n \in A$ s.t. n is the smallest element of A.

$n > 1$ by (1). $n-1 \in \mathbb{Z}^+$

By choice of n, $n-1 \notin A$

$S(n-1)$ is true.

By (2), $S((n-1)+1) = S(n)$ is true $\lightning$

Induction $\Rightarrow$ well-ordering

$\emptyset \neq A \subseteq \mathbb{Z}^+$. Suppose A does not contain a smallest element.

$n_0 \in A$

$n_1 \in A$, $\quad n_1 < n_0$

$n_2 \in A$, $\quad n_2 < n_1$ $\qquad\qquad n_k \leq n_0 - k$

$n_k \in A$, $\quad n_k < n_{k-1}$ $\qquad\qquad$ Put $k \geq n_0$.

$S(n) \to$ a statement about $n \geq n_0$.

(1) $S(n_0)$ is true

(2) $\forall\, n \geq n_0$, $S(n) \Rightarrow S(n+1)$

Then, $S(n)$ is true for all $n \geq n_0$.

$n_0 = 1$

$n_0 = 0, 2, 4, 100, \ldots$

Example 1 $\forall n \geq 0$, we have

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Proof: [basis] $n = 0$

empty sum $= 0$

$$\text{RHS} = \frac{0 \times 1}{2} = 0$$

[Induction] $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

$$1 + 2 + 3 + \cdots + n + (n+1) = \frac{n(n+1)}{2} + (n+1)$$

$$= \frac{(n+1)(n+2)}{2} = \frac{(n+1)((n+1)+1)}{2}$$

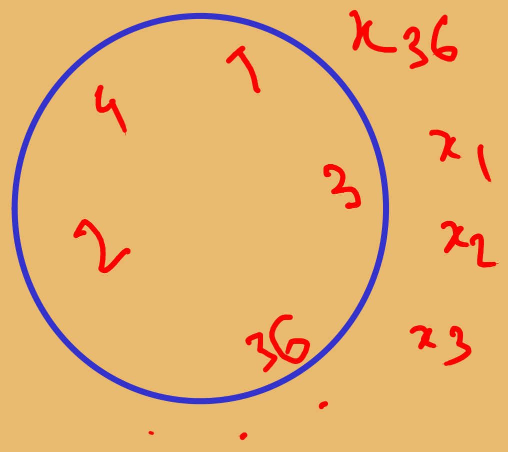
Example 2 (Application)

Three consecutive numbers

x_i, x_{i+1}, x_{i+2}

must exist

s.t. $x_i + x_{i+1} + x_{i+2} \geq 55.$



$$x_1 + x_2 + x_3 < 55$$
$$x_2 + x_3 + x_4 < 55$$
$$\overline{}$$
$$x_{36} + x_1 + x_2 < 55$$

$$\overline{3(1+2+3+\dots+36) < 36 \times 55}$$

$$3 \times \frac{\cancel{36} \times 37}{2} < \cancel{36} \times 55$$

$$111 < 110$$

Example 3 $\quad \forall n \geq 0, \quad n < 2^n.$

[Basis] $\quad n = 0$

$0 < 1 = 2^0$

[Induction] $\quad n < 2^n$ for some $n \geq 0$

$2^n \geq 2^0 = 1$

$1 \leq 2^n$

$$\frac{\begin{array}{c} n < 2^n \\ 1 \leq 2^n \end{array}}{n + 1 < 2^n + 2^n = 2^{n+1}.}$$

$H_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$, $n \geq 1$.

harmonic numbers

$\forall n \geq 1$, $H_1 + H_2 + \cdots + H_n = (n+1) H_n - n$

Proof: [Basis] $n = 1$

$LHS = H_1 = 1$

$RHS = (1+1) \times 1 - 1 = 1$

[Induction] $H_1 + H_2 + \cdots + H_n = (n+1) H_n - n$ for some $n \geq 1$

$$H_1 + H_2 + \cdots + H_n + H_{n+1} = (n+1) H_n - n + H_{n+1}$$
$$= (n+1)\left(H_{n+1} - \frac{1}{n+1}\right) - n + H_{n+1}$$
$$= (n+2) H_{n+1} - (n+1)$$
$$= ((n+1)+1) H_{n+1} - (n+1)$$

Fibonacci numbers

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad \forall\, n \geq 2.$$

$$\forall\, n \geq 0, \quad F_0^2 + F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$$

Proof: [Basis] $n = 0$

$$LHS = F_0^2 = 0^2 = 0$$

$$RHS = F_0 F_1 = 0 \times 1 = 0$$

[Induction] $F_0^2 + F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$ for some $n \geq 0$

$$F_0^2 + F_1^2 + \cdots + F_n^2 + F_{n+1}^2 = F_n F_{n+1} + F_{n+1}^2$$

$$= F_{n+1}(F_n + F_{n+1}) = F_{n+1} F_{n+2}$$

Ordered com|positions of $n \in \mathbb{N}$

$$\boxed{2^{n-1}}$$

$4 = \boxed{4}$

$= 3 + 1$

$= \boxed{1 + 3}$

$= \boxed{2 + 2}$

$= \boxed{1 + 1 + 2}$

$= 1 + 2 + 1$

$= 2 + 1 + 1$

$= 1 + 1 + 1 + 1$

[Basis] $n = 1$

$1 \qquad 2^{1-1} = 2^0 = 1$

[Induction] n has 2^{n-1} ordered compositions

To count the no. of ordered compositions of $n+1$

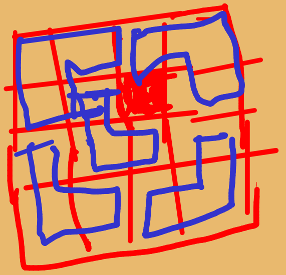
$n = 3$

$$
\begin{aligned}
3 &= 1+1+1 \\
&= 1+2 \\
&= 2+1 \\
&= 3
\end{aligned}
\qquad
\begin{aligned}
4 &= 1+1+1+1 &&= 1+1+2 \\
&= 1+2+1 &&= 1+3 \\
&= 2+1+1 &&= 2+2 \\
&= 3+1 &&= 4
\end{aligned}
$$

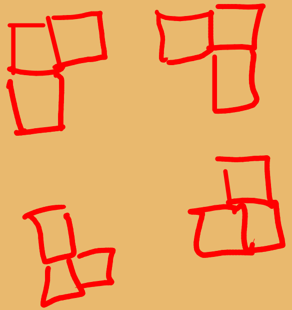
From the 2^{n-1} ordered compositions of n, we have two sets of ordered composition of $n+1$.

$$2^{n-1} + 2^{n-1} = 2^n = 2^{n+1-1}.$$

$2^n \times 2^n$ chessboard, $n \geq 1$.



Remove one cell

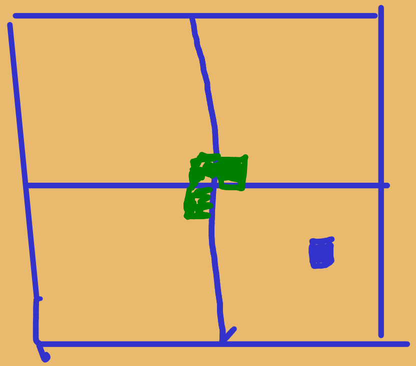


L's

$\frac{4^n - 1}{3}$ L's are given

[Basis] $n = 1$  2×2 board

[Induction] True for some $n \geq 1$.



$2^{n+1} \times 2^{n+1}$

Break into 4 $2^n \times 2^n$ subboards.