CS21001 Discrete Structures

Second Programming Assignment

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Solving Recurrence Relations

In this assignment, you write a program to find the solution of recurrence relations of the form

$$a_n = ra_{n-1} + sa_{n-2} + t,$$

where r, s, t and the first two terms a_0, a_1 of the sequence are supplied by the user as integer constants. We assume that $s \neq 0$. The characteristic equation of this recurrence is $x^2 - rx - s = 0$. You need to compute the roots of this equation, and handle all the cases like integer roots with possible repetitions, and non-integer (real or complex) roots occurring in conjugate pairs. Moreover, for the non-homogeneous part, you should consider all the possibilities of 1 being a root of the characteristic equation with multiplicity 0, 1, or 2. Since the theory of recurrence relations is covered in detail in the lectures, we concentrate only on the programming part and the input/output specifications of your program.

Part 1: Dealing with rationals and quadratic irrationals

We have solved the recurrence for Fibonacci numbers as $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$. We did not go for a floating-point approximation for $\sqrt{5}$. In this assignment, we will follow the same idea, and never go for floating-point approximations of rational or irrational numbers. Since we start with the roots of quadratic equations, we need to **exactly** represent real/complex numbers of the form

$$\frac{a + b\sqrt{d}}{c}$$

Here, c is a positive integer, whereas a, b, d are arbitrary integers. We enforce $\gcd(a, b, c) = 1$. Let us call these *special numbers*. Depending upon the value of d, there are different types of special numbers.

If $d > 0$ is not a perfect square, our special number is a quadratic irrational. The two roots of the characteristic equation are $\frac{r \pm \sqrt{r^2+4s}}{2}$. If r is even and $r^2 + 4s$ is a multiple of 4 but not a perfect square, we cancel one factor of 2 from the numerator and the denominator. We fix the value of d to $r^2 + 4s$ or $(r^2 + 4s)/4$ in all future computations. Do not attempt to process square factors further in it. For example, keep $\sqrt{45}$ as such, do not try to simplify it as $3\sqrt{5}$.

If $d \geqslant 0$ is a perfect square, then the two roots of the characteristic equation are actually integers. We absorb the contribution of $\sqrt{d}$ in a, and take $d = 0$. In this case, we deal with rational numbers of the form $\frac{a}{c}$.

Finally, if $d < 0$, we are dealing with complex numbers. We will keep the negative sign of d inside the radical as in $\frac{1+\sqrt{-3}}{2}$.

Define a data type **splnum** to store the four components a, b, c, d (all are integers) of special numbers. If you use C, you need to define a structure. If you use C++, you may use a structure or define a class. **(2)**

Write the following functions to implement the arithmetic of special numbers. If you use a C++ class for **splnum**, you may overload the standard arithmetic operators. You may add other functions (like negation, check for zero) if you find those convenient. **(6)**

```
splnum addspl ( splnum u, splnum v );   /* Returns u+v */
splnum subspl ( splnum u, splnum v );   /* Returns u−v */
splnum mulspl ( splnum u, splnum v );   /* Returns uv */
splnum invspl ( splnum u );             /* Returns 1/u provided that u is not zero */
splnum divspl ( splnum u, splnum v );   /* Returns u/v provided that v is not zero */
void prnspl ( splnum u );               /* Print u as in the sample outputs */
```

Write a function

```
void findroots ( int r, int s, splnum roots[2] );
```

to compute the two roots of the characteristic equation and store them in the array **roots**. **(4)**

Part 2: Solving the homogeneous recurrence **(8)**

Write a function

```
void solvehomogeneous ( int r, int s, int a0, int a1 );
```

to print the closed-form solution of the homogeneous recurrence $a_n = ra_{n-1} + sa_{n-2}$ with the two initial terms a_0, a_1 as specified in the call. This function should call **findroots** to compute the two roots ρ_1, ρ_2 of the characteristic equation. If $\rho_1 \neq \rho_2$, then the solution is $a_n = U\rho_1^n + V\rho_2^n$. If $\rho_1 = \rho_2 = \rho$, then the solution is $a_n = (Un+V)\rho^n$. The constants U, V are to be determined as **splnum**'s from the initial conditions.

Part 3: Solving the non-homogeneous recurrence **(12)**

Write a function

```
void solvenonhomogeneous ( int r, int s, int t, int a0, int a1 );
```

to print the closed-form solution of the homogeneous recurrence $a_n = ra_{n-1} + sa_{n-2} + t$ with the two initial terms a_0, a_1 as specified in the call. Compute and print the particular solution first. Then compute the homogeneous solution, and print it. Since $t = t \times 1^n$, you need to check whether 1 is a root of the characteristic equation, and if so, what its multiplicity is. The particular solution depends on that.

The main function

- Read r, s, t, a_0, a_1 from the user.
- Call **solvehomogeneous()** to print the solution of the homogeneous equation.
- Call **solvenonhomogeneous()** to print the solution of the nonhomogeneous equation.

Output **(8)**

Your code will be tested on several inputs. Some test cases are supplied below, Do not use global/static variables. Submit a single C/C++ file. Do not use STL. Your code must compile without any extra flag (except **-lm** for C programs).

```
**************** TEST CASE 1 (Fibonacci-type recurrence) ****************
r = 1
s = 1
t = 2
a0 = 0
a1 = 1

+++ Solving the homogeneous recurrence
    Characteristic equation: x^2 + (-1)x + (-1) = 0
    Root 1 = (1 + sqrt(5)) / 2
    Root 2 = (1 - sqrt(5)) / 2
    Homogeneous solution :
    [(sqrt(5)) / 5] [(1 + sqrt(5)) / 2]^n + [(-sqrt(5)) / 5] [(1 - sqrt(5)) / 2]^n

+++ Solving the nonhomogeneous recurrence
    Characteristic equation: x^2 + (-1)x + (-1) = 0
    Root 1 = (1 + sqrt(5)) / 2
    Root 2 = (1 - sqrt(5)) / 2
    Particular solution : -2
    Homogeneous solution :
    [(5 + 2 sqrt(5)) / 5] [(1 + sqrt(5)) / 2]^n + [(5 - 2 sqrt(5)) / 5] [(1 - sqrt(5)) / 2]^n
```

```
**************** TEST CASE 2 (Complex roots) ****************
r = 1
s = -7
t = -1
a0 = 1
a1 = 3

+++ Solving the homogeneous recurrence
    Characteristic equation: x^2 + (-1)x + (7) = 0
    Root 1 = (1 + sqrt(-27)) / 2
    Root 2 = (1 - sqrt(-27)) / 2
    Homogeneous solution :
    [(27 - 5 sqrt(-27)) / 54] [(1 + sqrt(-27)) / 2]^n + [(27 + 5 sqrt(-27)) / 54] [(1 - sqrt(-27)) / 2]^n

+++ Solving the nonhomogeneous recurrence
    Characteristic equation: x^2 + (-1)x + (7) = 0
    Root 1 = (1 + sqrt(-27)) / 2
    Root 2 = (1 - sqrt(-27)) / 2
    Particular solution : (-1) / 7
    Homogeneous solution :
    [(12 - 2 sqrt(-27)) / 21] [(1 + sqrt(-27)) / 2]^n + [(12 + 2 sqrt(-27)) / 21] [(1 - sqrt(-27)) / 2]^n

**************** TEST CASE 3 (Distinct integer roots) ****************
r = 2
s = 3
t = 3
a0 = 1
a1 = 2

+++ Solving the homogeneous recurrence
    Characteristic equation: x^2 + (-2)x + (-3) = 0
    Root 1 = 3
    Root 2 = -1
    Homogeneous solution :
    [3 / 4] [3]^n + [1 / 4] [-1]^n

+++ Solving the nonhomogeneous recurrence
    Characteristic equation: x^2 + (-2)x + (-3) = 0
    Root 1 = 3
    Root 2 = -1
    Particular solution : -3 / 4
    Homogeneous solution :
    [9 / 8] [3]^n + [5 / 8] [-1]^n

**************** TEST CASE 4 (Repeated roots) ****************
r = 4
s = -4
t = 3
a0 = 1
a1 = 1

+++ Solving the homogeneous recurrence
    Characteristic equation: x^2 + (-4)x + (4) = 0
    Root 1 = 2
    Root 2 = 2
    Homogeneous solution :
    [(-1 / 2)n + (1)] 2^n

+++ Solving the nonhomogeneous recurrence
    Characteristic equation: x^2 + (-4)x + (4) = 0
    Root 1 = 2
    Root 2 = 2
    Particular solution : 3
    Homogeneous solution :
    [(1)n + (-2)] [2]^n

**************** TEST CASE 5 (1 is a root of multiplicity 1) ****************
r = 4
s = -3
t = 1
a0 = 2
a1 = 7

+++ Solving the homogeneous recurrence
    Characteristic equation: x^2 + (-4)x + (3) = 0
    Root 1 = 3
    Root 2 = 1
```

```
    Homogeneous solution :
    [5 / 2] [3]^n + [-1 / 2] [1]^n

+++ Solving the nonhomogeneous recurrence
    Characteristic equation: x^2 + (-4)x + (3) = 0
    Root 1 = 3
    Root 2 = 1
    Particular solution : [-1 / 2] n
    Homogeneous solution :
    [11 / 4] [3]^n + [-3 / 4] [1]^n

**************** TEST CASE 6 (1 is a root of multiplicity 2) ****************
r = 2
s = -1
t = 1
a0 = 4
a1 = 7

+++ Solving the homogeneous recurrence
    Characteristic equation: x^2 + (-2)x + (1) = 0
    Root 1 = 1
    Root 2 = 1
    Homogeneous solution :
    [(3)n + (4)] 1^n

+++ Solving the nonhomogeneous recurrence
    Characteristic equation: x^2 + (-2)x + (1) = 0
    Root 1 = 1
    Root 2 = 1
    Particular solution : [1 / 2] n^2
    Homogeneous solution :
    [(5 / 2)n + (4)] 1^n
```