CS21001 Discrete Structures, Autumn 2020–2021

Long Test 3

Date: 17-Nov-2020 Maximum marks: 40

Instructions

- Answer all the questions. Be brief and precise.
- If you use any theorem/result/formula covered in lectures/tutorials, just mention, do not elaborate.

---

1. Let $a_n = \sum_{i=n}^{\infty} \frac{2^i}{i!}$ for all integers $n \geqslant 0$.

(a) Find a closed-form expression for the (ordinary) generating function $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots$ of the sequence $a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$. **(7)**

Solution We have

$$\begin{aligned}
A(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots \\
&= \left(\sum_{i=0}^{\infty} \frac{2^i}{i!}\right) + \left(\sum_{i=1}^{\infty} \frac{2^i}{i!}\right) x + \left(\sum_{i=2}^{\infty} \frac{2^i}{i!}\right) x^2 + \cdots + \left(\sum_{i=n}^{\infty} \frac{2^i}{i!}\right) x^n + \cdots \\
&= \sum_{n=0}^{\infty} \left(\sum_{i=n}^{\infty} \frac{2^i}{i!}\right) x^n \\
&= \sum_{n=0}^{\infty} \left[\left(1 + x + x^2 + x^3 + \cdots + x^n\right) \frac{2^n}{n!}\right] \\
&= \sum_{n=0}^{\infty} \left[\left(\frac{1 - x^{n+1}}{1 - x}\right) \frac{2^n}{n!}\right] \\
&= \frac{1}{1-x} \left[\sum_{n=0}^{\infty} \left(\frac{2^n}{n!}\right) - x \left(\sum_{n=0}^{\infty} \frac{(2x)^n}{n!}\right)\right] \\
&= \frac{e^2 - xe^{2x}}{1-x}.
\end{aligned}$$

(b) Use the expression for $A(x)$ in Part (a) to prove that $\sum_{n=0}^{\infty} a_n = 3e^2$. **No credit** if you do not use the closed-form expression for $A(x)$ derived in Part (a). **(3)**

Solution The desired sum is $A(1)$. But $A(1)$ is of the form $0/0$, so the desired sum is $\lim_{x \to 1} A(x)$, provided that the limit exists. By using l'Hôspital's rule, we get

$$\sum_{n=0}^{\infty} a_n = \lim_{x \to 1} A(x) = \lim_{x \to 1} \frac{e^2 - xe^{2x}}{1-x} = \lim_{x \to 1} \frac{-(e^{2x} + 2xe^{2x})}{-1} = 3e^2.$$

2. Let $m \geqslant 1$ be an integer constant. Let $b_n^{(m)}$ denote the number of ordered partitions (that is, compositions) of the integer $n \geqslant 0$ such that no summand is larger than m.

(a) Prove that the (ordinary) generating function of $b_n^{(m)}$ is **(7)**

$$B^{(m)}(x) = \frac{1-x}{1 - 2x + x^{m+1}}.$$

Solution Write n as a sum of i summands for some $i \geqslant 0$. Each summand is in the range $1, 2, 3, \ldots, m$, so the generating function for writing n as a sum of i summands in the given range is $(x + x^2 + x^3 + \cdots + x^m)^i$. Now, we vary i from 0 to ∞ to get

$$B^{(m)}(x) = \sum_{i=0}^{\infty} (x + x^2 + x^3 + \cdots + x^m)^i = \frac{1}{1 - (x + x^2 + x^3 + \cdots + x^m)} = \frac{1}{1 - x\left(\frac{1 - x^m}{1 - x}\right)} = \frac{1-x}{1 - 2x + x^{m+1}}.$$

(b) From the formula of Part (a), deduce that $b_n^{(2)} = F_{n+1}$, where $F_0, F_1, F_2, \ldots$ is the sequence of Fibonacci numbers. **No credit** for using any argument other than using the formula of Part (a). **(3)**

Solution Let $F(x)$ denote the generating function of the Fibonacci sequence. Putting $m = 2$ and $F_0 = 0$ gives

$$
\begin{aligned}
B^{(2)}(x) &= \frac{1-x}{1-2x+x^3} = \frac{1}{1-x-x^2} = \frac{1}{x}\left(\frac{x}{1-x-x^2}\right) = \frac{1}{x}F(x) \\
&= \frac{1}{x}\left[F_0 + F_1 x + F_2 x^2 + F_3 x^3 + \cdots + F_{n+1}x^{n+1} + \cdots\right] \\
&= F_1 + F_2 x + F_3 x^2 + \cdots + F_{n+1}x^n + \cdots.
\end{aligned}
$$

From this expression, the result follows immediately.

3. Solve the following recurrence relation, and deduce the closed-form expression for $T(n)$.

$$
T(n) = \begin{cases} \sqrt{n}\,T(\sqrt{n}) + n(\log_2 n)^d, & \text{if } n > 2 \\ 2 \qquad , & \text{if } n = 2 \end{cases} \quad (d \geqslant 0).
$$

Note: Use of generating functions is **not** allowed to solve this problem. **(10)**

Solution Given that $T(n) = \sqrt{n}\,T(\sqrt{n}) + n\log_2^d n$ (where $d \geqslant 0$) and $T(2) = 2$, we have:

$$
\begin{aligned}
\frac{T(n)}{n} &= \frac{T(\sqrt{n})}{\sqrt{n}} + \log_2^d n && \ldots\left[\textit{dividing both sides by } n\right] \\
\Rightarrow \quad S(n) &= S(\sqrt{n}) + \log_2^d n && \ldots\left[\textit{assuming } S(n) = \frac{T(n)}{n}\right] \\
\Rightarrow \quad S(2^{2^k}) &= S(2^{2^{(k-1)}}) + (2^k)^d && \ldots\left[\textit{substituting } n = 2^{2^k}\right] \\
\Rightarrow \quad R(k) &= R(k-1) + (2^k)^d && \ldots\left[\textit{let } R(k) = S(2^{2^k})\right] \\
\Rightarrow \quad R(k) &= R(0) + (2^d)^1 + (2^d)^2 + \cdots + (2^d)^{k-1} + (2^d)^k && \ldots\left[\textit{because } (2^k)^d = 2^{kd} = (2^d)^k\right] \\
\Rightarrow \quad R(k) &= 1 + \sum_{i=1}^{k} (2^d)^i && \ldots\left[S(2) = \frac{T(2)}{2} = 1, \textit{ implying } R(0) = S(2^{2^0}) = 1\right] \\
\Rightarrow \quad R(k) &= \begin{cases} \frac{(2^d)^{(k+1)}-1}{2^d-1}, & \text{if } d > 0 \\ 1+k, & \text{if } d = 0 \end{cases}
\end{aligned}
$$

Hence, $R(k) = S(2^{2^k}) = \begin{cases} \frac{(2^d)^{(k+1)}-1}{2^d-1}, & \text{if } d > 0 \\ 1+k, & \text{if } d = 0 \end{cases}$ where $n = 2^{2^k}$ and $S(n) = \dfrac{T(n)}{n}$,

which means $S(n) = \begin{cases} \frac{2^d \log_2^d n - 1}{2^d - 1}, & \text{if } d > 0 \\ 1 + \log_2 \log_2 n, & \text{if } d = 0 \end{cases}$ implying $T(n) = \begin{cases} \frac{2^d n \log_2^d n - n}{2^d - 1}, & \text{if } d > 0 \\ n + n\log_2 \log_2 n, & \text{if } d = 0 \end{cases}$

Marking Scheme

- Substitutions and Simplifications: 4 marks
- Complete Deduction Steps: 4 marks
- Final closed-form: 2 marks
- *Deduct* 2 marks if $d = 0$ case not considered overall

4. Solve the following recurrence relation, and deduce the closed-form expression for a_n.

$$
a_n = a_{n-1} + 8a_{n-2} - 12a_{n-3} + 2^n \text{ (for } n \geqslant 3), \quad \text{with } a_0 = 1,\ a_1 = 1,\ a_2 = \frac{83}{5}.
$$

Note: Use of generating functions is **not** allowed to solve this problem. **(10)**

Solution Given the recurrence $a_n = a_{n-1} + 8a_{n-2} - 12a_{n-3} + 2^n$ (for $n \geqslant 2$), we find that the homogeneous part is

$$a_n = a_{n-1} + 8a_{n-2} - 12a_{n-3}.$$

So, the *characteristic equation* that results from the homogeneous part will be $x^3 - x^2 - 8x + 12 = 0$.

Solving for x, we get

$$x^3 - x^2 - 8x + 12 = 0 \quad \Rightarrow \quad (x-2)^2(x+3) = 0 \quad \Rightarrow \quad x = 2 \text{ (double root)}, \; x = -3.$$

Therefore, the homogeneous solution is $a_n^{(h)} = (A + Bn)2^n + C(-3)^n$.

Now, the particular solution (with respect to 2^n) will be $a_n^{(p)} = Dn^2 2^n$ $\big($due to conflict term $(A+Bn)2^n$ in $a_n^{(h)}\big)$.

Solving for constant D with the help of the given recurrence, we get

$$\begin{aligned} & Dn^2 2^n && = D(n-1)^2 2^{n-1} + 8D(n-2)^2 2^{n-2} - 12D(n-3)^2 2^{n-3} + 2^n \\ \Rightarrow \quad & Dn^2 && = \frac{D}{2}(n-1)^2 + 2D(n-2)^2 - \frac{3}{2}D(n-3)^2 + 1. \end{aligned}$$

Comparing the constant terms (coefficients of n^0) in above equation, we find

$$0 = \frac{D}{2} + 8D - \frac{27}{2} + 1 \quad \Rightarrow \quad D = \frac{1}{5}.$$

So, the final (parameterized) solution is

$$a_n = a_n^{(h)} + a_n^{(p)} = (A + Bn)2^n + C(-3)^n + \frac{1}{5}n^2 2^n.$$

Solving for the constants A, B, C, we get

$$\begin{aligned} a_0 = 1 = A + C \quad & \Rightarrow \quad C = 1 - A, \\ a_1 = 1 = 2(A+B) - 3C + \frac{2}{5} \quad & \Rightarrow \quad 5A + 2B = \frac{18}{5}, \\ a_2 = \frac{83}{5} = 4(A+2B) + 9C + \frac{16}{5} \quad & \Rightarrow \quad -5A + 8B = \frac{22}{5}. \end{aligned}$$

The above three equations produce $A = \frac{2}{5}$, $B = \frac{4}{5}$, $C = \frac{3}{5}$.

Hence, the final solution is

$$a_n \quad = \quad \left(\frac{2}{5} + \frac{4}{5}n\right)2^n + \frac{3}{5}(-3)^n + \frac{1}{5}n^2 2^n \quad = \quad \frac{1}{5}\Big[(1+2n)2^{n+1} + (-1)^n 3^{n+1} + n^2 2^n\Big].$$

Marking Scheme

- Characteristic equation formation and roots determination: 2 marks
- Homogeneous solution formation: 1 mark
- Particular solution formation: 1 mark
- Constant Solve for particular part: 2 marks
- Constant Solve for homogeneous part: 3 marks
- Final solution: 1 mark