CS21001 Discrete Structures, Autumn 2020–2021

Long Test 2

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1. Let a be a positive integer which is not a multiple of 3. You initialize the number $n = a^{2020}$, and enter the following loop. Assume that we use a special-purpose multiple-precision integer library to hold all the digits of large integers exactly.

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while (n ⩾ 10^10) {
    Set r = n % 10;   /* Remainder of Euclidean division by 10 */
    Set n = n / 10;   /* Quotient of Euclidean division by 10 */
    Set n = n + r;
}
```

The loop terminates when n becomes smaller than 10^{10}. Treat this n as a ten-digit number (insert leading zero digits if necessary). Prove that among these ten digits, at least one digit must be repeated. **(10)**

Solution Let $\rho = a^{2020} \text{ rem } 3$. Since a is not a multiple of 3, we have $\rho \neq 0$. If $n = 10q + r$ at the top of the loop, the loop body changes n to $q + r$. We have $10q + r - (q + r) = 9q$ which is a multiple of 3. Thus at all times when the loop condition is checked, we have $n \equiv \rho \not\equiv 0 \pmod 3$. In particular, when the loop terminates, $n = (d_9 d_8 \ldots d_2 d_1 d_0)_{10}$ is still ρ modulo 3. Since $10 \equiv 1 \pmod 3$, we have $n = 10^9 d_9 + 10^8 d_8 + \cdots + 10^2 d_2 + 10 d_1 + d_0 \equiv d_9 + d_8 + \cdots + d_2 + d_1 + d_0 \pmod 3$. If there is no repeated digit in n at this point of time, $d_9, d_8, \ldots, d_2, d_1, d_0$ must be a permutation of $0, 1, 2, 3 \ldots, 9$, so $d_9 + d_8 + \cdots + d_2 + d_1 + d_0 \equiv 0 + 1 + 2 + 3 + \cdots + 9 \equiv 45 \equiv 0 \pmod 3$. But then $\rho = 0$, a contradiction.

2. Show that there exists an integer n such that $0 < \sin n < 2^{-2020}$. **(10)**

Solution Let $\varepsilon = \sin^{-1}(2^{-2020})$. Take an integer $n > 1/\varepsilon$. We know that π is an irrational number. Therefore $k\pi$ is never an integer for any integer $k \neq 0$. Break the interval $[0, 1)$ into n subintervals $[0, \frac{1}{n})$, $[\frac{1}{n}, \frac{2}{n})$, $[\frac{2}{n}, \frac{3}{n})$, $\ldots$, $[1 - \frac{1}{n}, 1)$. Consider the $n+1$ real numbers $2\pi, 4\pi, 6\pi, \ldots, 2(n+1)\pi$. Let $\{2i\pi\}$ denote the fractional part of $2i\pi$. Since there are $n+1$ such proper fractions, two of them must belong to the same subinterval of length $1/n$. Let these be $\{2i\pi\}$ and $\{2j\pi\}$ with $\{2i\pi\} > \{2j\pi\}$. Let $\delta = \{2i\pi\} - \{2j\pi\}$. Then $0 < \delta < 1/n < \varepsilon$. We have $2i\pi = s + \{2i\pi\}$ and $2j\pi = t + \{2j\pi\}$ for some integers s and t. Take $n = t - s$. We have $n = t - s = 2(j - i)\pi + \delta$, so $\sin n = \sin[2(j - i)\pi + \delta] = \sin \delta$. Since sin is a strictly increasing function in the range $[0, \pi/2)$, and since $0 < \delta < \varepsilon$, we have $0 = \sin 0 < \sin \delta < \sin \varepsilon = 2^{-2020}$.

3. Let $\mathbb{Z}$ be the set of all integers. Define a relation R on $\mathbb{N}$ (the the set of positive integers) as follows:

$$\forall a, b \in \mathbb{N},\ a\,R\,b \text{ if and only if } \exists i \in \mathbb{Z},\ \frac{a}{b} = 2^i.$$

(a) Prove that R is an equivalence relation. **(5)**

Solution [Reflexive] For $a \in \mathbb{N}$, $\frac{a}{a} = 1 = 2^0$, so $a\,R\,a$.

[Symmetric] Suppose that $a\,R\,b$. Then, we have

$$\begin{aligned} a\,R\,b &\Rightarrow \exists i \in \mathbb{Z},\ \frac{a}{b} = 2^i \\ &\Rightarrow \exists i \in \mathbb{Z},\ \frac{b}{a} = 2^{-i} \\ &\Rightarrow b\,R\,a, \quad \text{because } -i \in \mathbb{Z}. \end{aligned}$$

[Transitive] Suppose that $a\,R\,b$ and $b\,R\,c$. By definition,

$$\begin{aligned} a\,R\,b &\Rightarrow \exists i \in \mathbb{Z},\ \frac{a}{b} = 2^i, \\ b\,R\,c &\Rightarrow \exists j \in \mathbb{Z},\ \frac{b}{c} = 2^j. \end{aligned}$$

But then $\frac{a}{c} = \frac{a}{b} \times \frac{b}{c} = 2^{i+j}$. Since $i + j \in \mathbb{Z}$, we have $a\,R\,c$.

(b) List the equivalence classes defined by R on $\mathbb{N}$. **(3)**

Solution Given any natural number x, there exist $k \in \mathbb{N}$ and $i \in \mathbb{N}_0$ such that k is not divisible by 2, and $x = k \times 2^i$. For any natural number $y = k \times 2^j$, we have $\frac{x}{y} = 2^{i-j}$, where $(i-j)$ is an integer. Thus, $x\,R\,y$, implying that x and y belong to the same equivalence class. Conversely, if $x = k \times 2^i$ and $y = l \times 2^j$ for odd k, l with $k \neq l$, then $\frac{x}{y}$ is not of the form 2^t with $t \in \mathbb{Z}$. Therefore the equivalence classes of R are as follows:

$$\begin{gathered}
\{1 \times 2^i \mid i = 0, 1, 2, \ldots\} \\
\{3 \times 2^i \mid i = 0, 1, 2, \ldots\} \\
\{5 \times 2^i \mid i = 0, 1, 2, \ldots\} \\
\{7 \times 2^i \mid i = 0, 1, 2, \ldots\} \\
\vdots
\end{gathered}$$

(c) Prove/Disprove: R is a partial order. **(2)**

Solution R is not a partial order as *anti-symmetry* does not hold. As a counter-example, $\frac{4}{2} = 2^1$ and $\frac{2}{4} = 2^{-1}$. However, $2 \neq 4$.

4. Let A be an infinite set. Prove that there exists a map $f : A \to A$ which is surjective (onto) but not injective (one-one). **(10)**

Solution Since A is infinite, it contains a countably infinite subset $B = \{b_1, b_2, \ldots, b_n, \ldots\}$. Define the map $f : A \to A$ as

$$f(a) = \begin{cases} b_1 & \text{if } a = b_1, \\ b_{n-1} & \text{if } a = b_n \text{ for } n \geqslant 2, \\ a & \text{if } a \notin B. \end{cases}$$

Clearly, f is surjective. But f is not injective, since $f(b_1) = f(b_2) = b_1$.
