CS21001 Discrete Structures, Autumn 2020–2021

Long Test 1

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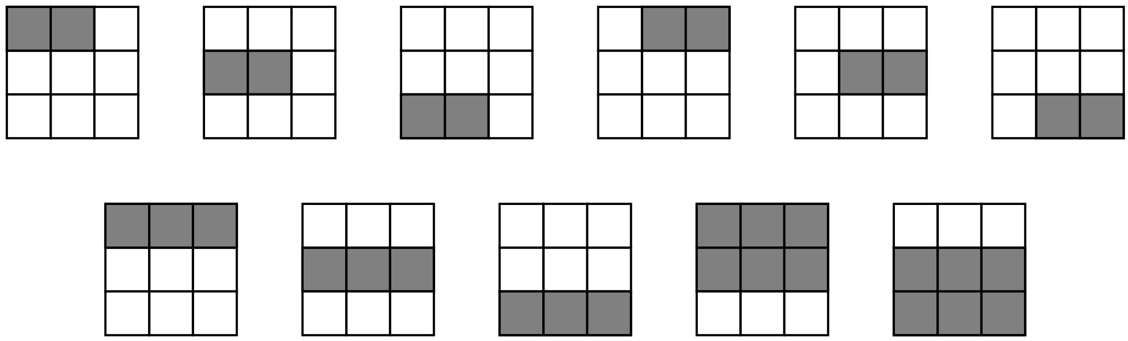
1. You are given an $n \times n$ grid. You want to count the number of rectangles in the grid, in which the horizontal sides are larger than the vertical sides. The figure below shows the 11 possible such rectangles in a 3×3 grid. Deduce a closed-form expression for the desired count (as a function of n). **(10)**



Solution Let $R_>$, $R_=$ and $R_<$ denote the numbers of rectangles in the grid, whose horizontal sides are respectively larger than, equal to, and smaller than the vertical sides. The total number of rectangles is

$$R = R_> + R_= + R_<.$$

Here, squares too are treated as rectangles.

We first deduce a formula for R. A rectangle in the grid is specified fully by the coordinates (i, j) of the bottom left corner and the coordinates (k, l) of the top right corner. We must have $0 \leqslant i < k \leqslant n$ and $0 \leqslant j < l \leqslant n$. This implies that i and k can be chosen in $\binom{n+1}{2}$ ways, and j and l can also be chosen in $\binom{n+1}{2}$ ways. Therefore the total number of rectangles is

$$R = \binom{n+1}{2}^2 = \frac{n^2(n+1)^2}{4}.$$

Then, we deduce a formula for $R_=$. There are n^2 squares of side 1, $(n-1)^2$ squares of side 2, $(n-2)^2$ squares of side 3, $\ldots$, 1 square of side n. Therefore

$$R_= = n^2 + (n-1)^2 + (n-2)^2 + \cdots + 2^2 + 1^2 = \frac{n(n+1)(2n+1)}{6}.$$

We finally note that by symmetry

$$R_> = R_<.$$

Therefore the desired count is

$$\begin{aligned}
R_> &= \frac{1}{2}\Big(R - R_=\Big) \\
&= \frac{n(n+1)}{24}\Big(3n(n+1) - 2(2n+1)\Big) \\
&= \frac{n(n+1)}{24}\Big(3n^2 - n - 2\Big) \\
&= \frac{n(n+1)(n-1)(3n+2)}{24}.
\end{aligned}$$

2. Your task is to (logically) solve a murder mystery on behalf of Sherlock Holmes, which appeared in the novel "A Study in Scarlet" by Sir Arthur Conan Doyle. The arguments (simplified from the novel) go as follows.

1. There was a murder. If it was not done for robbery, then either it was a political assassination, or it might be for a woman.
2. In case of robbery, usually something is taken.
3. However, nothing was taken from the murderer's place.
4. Political assassins leave the place immediately after their assassination work gets completed.
5. On the contrary, the assassin left his/her tracks all over the murderer's place.
6. For an assassin, to leave tracks all over the murderer's place indicates that (s)he was there all the time (for long duration).

Please logically frame and derive the solution. Present your answer as asked in the following parts.

(a) Write all propositions with English meaning (statements) that you have used. **(2)**

Solution We may use the following propositions.

rob	:	The murder was done for robbery.
pol	:	The murder was a political assassination.
wom	:	The murder was for a woman.
tak	:	Something was taken from the murderer's place.
imm	:	The assassin left immediately after work done.
trc	:	The assassin left tracks all over the room.

(b) Build suitable propositional logic formula to encode each of the *six* statements above. **(3)**

Solution (1) $\neg\text{rob} \to \text{pol} \vee \text{wom}$
(2) $\text{rob} \to \text{tak}$
(3) $\neg\text{tak}$
(4) $\text{pol} \to \text{imm}$
(5) trc
(6) $\text{trc} \to \neg\text{imm}$

(c) Show all deduction steps (with the name of the rules you apply) to derive the goal (solve the mystery). **(4)**

Solution $\text{rob} \to \text{tak}$
$\neg\text{tak}$
――――――― (Modus Tollens)
$\therefore \neg\text{rob}$

$\neg\text{rob}$
$\neg\text{rob} \to \text{pol} \vee \text{wom}$
――――――― (Modus Ponens)
$\therefore \text{pol} \vee \text{wom}$

trc
$\text{trc} \to \neg\text{imm}$
――――――― (Modus Ponens)
$\therefore \neg\text{imm}$

$\neg\text{imm}$
$\text{pol} \to \text{imm}$
――――――― (Modus Tollens)
$\therefore \neg\text{pol}$

$\text{pol} \vee \text{wom}$
$\neg\text{pol}$
――――――― (Disjunctive Syllogism)
$\therefore \text{wom}$

(d) Conclude what the reason for the murder was. **(1)**

Solution The murder was done for a woman.

3. Let

$$H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

denote the n-th harmonic number. By induction on n, prove that

$$H_n \leqslant 1 + \ln n$$

for all $n \geqslant 1$. **No credit** if any method other than induction is used. **(10)**

Solution [Base] For $n = 1$, we have $H_1 = 1 \leqslant 1 + 0 = 1 + \ln 1$.

[Induction] Take $n \geqslant 1$, and assume that $H_n \leqslant 1 + \ln n$. Then we have

$$\begin{aligned}
H_{n+1} &= H_n + \frac{1}{n+1} \\
&\leqslant 1 + \ln n + \frac{1}{n+1} \\
&= 1 + \ln(n+1) + \frac{1}{n+1} + (\ln n - \ln(n+1)) \\
&= 1 + \ln(n+1) + \frac{1}{n+1} + \ln\left(\frac{n}{n+1}\right) \\
&= 1 + \ln(n+1) + \frac{1}{n+1} + \ln\left(1 - \frac{1}{n+1}\right) \\
&= 1 + \ln(n+1) + \frac{1}{n+1} - \frac{1}{n+1} - \frac{1}{2}\left(\frac{1}{n+1}\right)^2 - \frac{1}{3}\left(\frac{1}{n+1}\right)^3 - \frac{1}{4}\left(\frac{1}{n+1}\right)^4 - \cdots \\
&= 1 + \ln(n+1) - \left[\frac{1}{2}\left(\frac{1}{n+1}\right)^2 + \frac{1}{3}\left(\frac{1}{n+1}\right)^3 + \frac{1}{4}\left(\frac{1}{n+1}\right)^4 + \cdots\right] \\
&\leqslant 1 + \ln(n+1).
\end{aligned}$$

Since $n \geqslant 1$, we have $0 < \frac{1}{n+1} \leqslant \frac{1}{2} < 1$, and so we can use the above expansion of $\ln\left(1 - \frac{1}{n+1}\right)$.

4. Consider the following recursive function.

```
int f ( int n, int i, int s, int t )
{
    if (i == n) return s * (n - 1) + 1;
    return f (n, i + 1, s + t, t * n);
}
```

You call **`f(n,0,0,1)`** from **`main()`** for a positive integer n. What does the call return as a function of n? Justify your answer. **(10)**

Solution The return value is n^n. In order to see why, we claim that for all $i \in \{0,1,2,\ldots,n\}$, the call **`f(n,i,s,t)`** receives the arguments n, i, $s = 1+n+n^2+\cdots+n^{i-1}$, and $t = n^i$. We may prove this claim by induction on i. In the outermost call (induction basis), we have $i = 0$, so s is the empty sum (0), and $t = n^0 = 1$. Now suppose that the result holds for some $i \geqslant 0$. If $i < n$, the function updates i to $i+1$, s to $s+t = 1+n+n^2+\cdots+n^{i-1}+n^i$, and t to $tn = n^i \times n = n^{i+1}$, for passing to the next recursive call. Finally, for $i = n$, the function makes no further recursive calls, but computes and returns the quantity

$$s \times (n-1)+1 = (1+n+n^2+\cdots+n^{n-1})(n-1)+1 = \left(\frac{n^n-1}{n-1}\right)(n-1)+1 = (n^n-1)+1 = n^n.$$

All other calls return the same value without any further processing.