CS21001 Discrete Structures, Autumn 2019–2020

Class Test 2

04–November–2019 | F116/F142, 06:15pm–07:15pm | Maximum marks: 20

Roll no: ________________ **Name:** ____

[*Write your answers in the question paper itself. Be brief and precise. Answer all questions. If you use any theorem/result/formula covered in the class, just mention it, do not elaborate.*]

1. Solve the recurrence relation $a_n = 3a_{n-1} - 2a_{n-2} + 2n - 7$ for $n \geqslant 2$ with the initial conditions $a_0 = 2$ and $a_1 = 5$. Show all the steps of your calculation. **(8)**

Solution The characteristic equation of the recurrence is

$$x^2 - 3x + 2 = 0, \text{ that is, } (x-1)(x-2) = 0,$$

so the characteristic roots of the recurrence are $1, 2$, and the homogeneous solution is

$$a_n^{(h)} = A \times 1^n + B \times 2^n = A + B \times 2^n.$$

The non-homogeneous part of the recurrence is $2n - 7 = (2n - 7) \times 1^n$, and 1 is a characteristic root of multiplicity one, so the particular solution is of the form

$$a_n^{(p)} = n(un + v).$$

Substituting this in the recurrence relation gives

$$n(un + v) = 3(n-1)(u(n-1)+v) - 2(n-2)(u(n-2)+v) + 2n - 7.$$

Simplification gives

$$(2u+2)n + (v - 5u - 7) = 0,$$

that is, $2u + 2 = 0$ and $v - 5u - 7 = 0$, that is $u = -1$ and $v = 5u + 7 = 2$, that is,

$$a_n^{(p)} = -n^2 + 2n.$$

It follows that

$$a_n = a_n^{(h)} + a_n^{(p)} = A + B \times 2^n - n^2 + 2n.$$

Now, plug in the initial conditions to get the system

$$\begin{aligned} a_0 &= 2 = A + B, \\ a_1 &= 5 = A + 2B + 1, \end{aligned}$$

which has the solution

$$A = 0 \text{ and } B = 2.$$

Therefore the solution of the recurrence relation is

$$a_n = 2^{n+1} - n^2 + 2n \text{ for all } n \geqslant 0.$$

2. Let $R = \mathbb{Z} \times \mathbb{Z}$ (Cartesian product). Define addition and multiplication on R as:

$$\begin{aligned} (a,b)+(c,d) &= (a+c,\ b+d), \\ (a,b)\cdot(c,d) &= (ac+ad+bc,\ 2ac+bd). \end{aligned}$$

(a) Verify that R is a ring under these two operations. **(6)**

Solution All ring axioms are now verified one by one.

[Closure under $+$] For $a,b,c,d \in \mathbb{Z}$, we have $a+c, b+d \in \mathbb{Z}$.

[Associativity of $+$] $(a+c)+e = a+(c+e)$ and $(b+d)+f = b+(d+f)$.

[Commutativity of $+$] $a+c = c+a$ and $b+d = d+b$.

[Additive identity] $(0,0)$ is the additive identity.

[Additive inverse] $-(a,b) = (-a,-b)$.

[Closure under $\cdot$] $ac+ad+bc,\ 2ac+bd \in \mathbb{Z}$.

[Associativity of $\cdot$] On the one hand, we have

$$\begin{aligned} & ((a,b)\cdot(c,d))\cdot(e,f) \\ = \ & (ac+ad+bc,\ 2ac+bd)\cdot(e,f) \\ = \ & ((ac+ad+bc)e+(ac+ad+bc)f+(2ac+bd)e,\ 2(ac+ad+bc)e+(2ac+bd)f) \\ = \ & (3ace+acf+ade+adf+bce+bcf+bde,\ 2ace+2acf+2ade+2bce+bdf). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & (a,b)\cdot((c,d)\cdot(e,f)) \\ = \ & (a,b)\cdot(ce+cf+de,\ 2ce+df) \\ = \ & (a(ce+cf+de)+a(2ce+df)+b(ce+cf+de),\ 2a(ce+cf+de)+b(2ce+df)) \\ = \ & (3ace+acf+ade+adf+bce+bcf+bde,\ 2ace+2acf+2ade+2bce+bdf). \end{aligned}$$

[Distributivity of $\cdot$ over $+$] We have

$$\begin{aligned} & (a,b)\cdot((c,d)+(c',d')) \\ = \ & (a,b)\cdot(c+c',d+d') \\ = \ & (a(c+c')+a(d+d')+b(c+c'),\ 2a(c+c')+b(d+d')) \\ = \ & (ac+ad+bc,\ 2ac+bd)+(ac'+ad'+bc',\ 2ac'+bd') \\ = \ & (a,b)\cdot(c,d)+(a,b)\cdot(c',d'). \end{aligned}$$

Likewise, show that $((a,b)+(a',b'))\cdot(c,d) = (a,b)\cdot(c,d)+(a',b')\cdot(c,d)$.

(b) Prove that R is a commutative ring with unity. **(3)**

Solution We have $(a,b)\cdot(c,d) = (ac+ad+bc, 2ac+bd)$ and $(c,d)\cdot(a,b) = (ca+cb+da, 2ca+db)$. Next, use the commutativity of integer addition and multiplication.

The multiplicative identity is $(0,1)$ since $(0,1)\cdot(a,b) = (0\times a+1\times a+0\times b, 2\times 0\times a+1\times b) = (a,b)$ and $(a,b)\cdot(0,1) = (a\times 0+a\times 1+b\times 0, 2a\times 0+b\times 1) = (a,b)$.

(c) Prove/Disprove: R is an integral domain. **(3)**

Solution *False*. $(1,1)(1,-2) = (1\times 1+1\times(-2)+1\times 1,\ 2\times 1\times 1+1\times(-2)) = (0,0)$.