CS21001 Discrete Structures, Autumn 2019–2020

Class Test 1

03–September–2019 | F116/F142, 06:15pm–07:15pm | Maximum marks: 20

Roll no: ____________ **Name:** ______________________________

[*Write your answers in the question paper itself. Be brief and precise. Answer all questions. If you use any algorithm/result/formula covered in the class, just mention it, do not elaborate.*]

1. Consider the following predicates with the specified meaning.

- $\text{owns}(x, y)$: x owns y
- $\text{bookLover}(x)$: x is a book lover
- $\text{mutilates}(x, y)$: x mutilates y
- $\text{book}(x)$: x is a book
- $\text{kindle}(x)$: x is a kindle

Encode the following statements in the first-order logic using the above predicates. **(1 × 5)**

(a) Tom owns a kindle.

Solution $\exists x \left[\text{owns}(\text{Tom}, x) \wedge \text{kindle}(x)\right]$

(b) Every kindle owner loves books.

Solution $\forall x\, \forall y \left[\left(\text{owns}(x, y) \wedge \text{kindle}(y)\right) \rightarrow \text{bookLover}(x)\right]$

(c) No book lover mutilates books.

Solution $\forall x\, \forall y \left[\left(\text{bookLover}(x) \wedge \text{book}(y)\right) \rightarrow \neg\, \text{mutilates}(x, y)\right]$

(d) Either Tom or Austin mutilates the book called Origin.

Solution $\text{book}(\text{Origin}) \wedge \left(\text{mutilates}(\text{Tom}, \text{Origin}) \vee \text{mutilates}(\text{Austin}, \text{Origin})\right)$

(e) Every kindle is a book.

Solution $\forall x \left[\text{kindle}(x) \rightarrow \text{book}(x)\right]$

2. Let A, B be two sets, and let S denote the set of all functions $A \to B$. Define a relation ρ on S as $f \,\rho\, g$ if and only if $f(A) = g(A)$ (that is, the images of f and g are the same subset of B).

(a) Prove that ρ is an equivalence relation on S. **(3)**

Solution [Reflexive] $\forall f \left[f(A) = f(A) \right].$

[Symmetric] $\forall f \, \forall g \left[\big(f(A) = g(A)\big) \to \big(g(A) = f(A)\big) \right]$

[Transitive] $\forall f \, \forall g \, \forall h \left[\big(f(A) = g(A)\big) \land \big(g(A) = h(A)\big) \to \big(f(A) = h(A)\big) \right]$

(b) Let S/ρ denote the set of all equivalence classes of ρ, and $\mathscr{P}(B)$ the power set of B. Define a function $F : S/\rho \to \mathscr{P}(B)$ by $F([f]_\rho) = f(A)$. Prove that F is well-defined and injective. **(2)**

Solution For all $f, g \in S$, we have

$$[f]_\rho = [g]_\rho \iff f(A) = g(A) \iff F([f]_\rho) = F([g]_\rho).$$

The forward implications establish well-defined-ness, whereas the backward implications establish injectivity.

3. Let a, b, c, d be four integers, distinct from one another. Prove that it is impossible to construct a polynomial $p(x)$ with *integer* coefficients such that $p(a) = p(b) = p(c) = 5$ and $p(d) = 6$. **(5)**

Solution Suppose that such a polynomial $p(x)$ can be constructed. For all $u, v \in \mathbb{Z}$ and $n \in \mathbb{N}_0$, we have $(u-v)|(u^n - v^n)$, and so $(u-v)|(p(u)-p(v))$. But then, the non-zero differences $d-a, d-b, d-c$ all divide $6-5=1$, whereas 1 has only two divisors ± 1. Therefore, at least two of the three differences $d-a, d-b, d-c$ must be the same, contradicting the fact that a, b, c are distinct from one another.

4. Specify the return value $W(n)$ of the following function as a closed-form expression in its input n (a non-negative integer). Prove your assertion. Assume the availability of a function `binomial(n,k)` that returns the floating-point value of the binomial coefficient $C(n,k) = \binom{n}{k}$. **(5)**

```
int W ( unsigned int n )
{
   double sum = 0, denominator = 1;
   int k;

   for (k=0; k<=n; ++k) {
      sum += binomial(n+k,k) / denominator;
      denominator *= 2;
   }
   return round(sum);
}
```

Solution We prove by induction on n that for all $n \geqslant 0$, we have

$$W(n) = \sum_{k=0}^{n} \left[\binom{n+k}{k} \frac{1}{2^k} \right] = 2^n.$$

[Basis] For $n = 0$, we have $W(0) = \binom{0}{0} = 1 = 2^0$.

[Induction] Assume that $n \geqslant 0$, and that $W(n) = 2^n$. By Pascal's identity, we have:

$$\begin{aligned}
W(n+1) &= \sum_{k=0}^{n+1} \left[\binom{n+1+k}{k} \frac{1}{2^k} \right] \\
&= \sum_{k=0}^{n+1} \left[\binom{n+k}{k} + \binom{n+k}{k-1} \right] \frac{1}{2^k} \\
&= \sum_{k=0}^{n} \left[\binom{n+k}{k} \frac{1}{2^k} \right] + \binom{n+n+1}{n+1} \frac{1}{2^{n+1}} + \sum_{k=1}^{n+1} \left[\binom{n+k}{k-1} \frac{1}{2^k} \right] + \binom{n}{-1} \\
&= W(n) + \binom{2n+1}{n+1} \frac{1}{2^{n+1}} + \frac{1}{2} \sum_{k=0}^{n} \left[\binom{n+1+k}{k} \frac{1}{2^k} \right],
\end{aligned}$$

where the last equality follows from replacing $k-1$ by k in the second sum. Now,

$$\binom{2n+1}{n+1} = \frac{(2n+1)!}{(n+1)!\,n!} = \frac{1}{2} \left[\frac{(2n+2)!}{(n+1)!\,(n+1)!} \right] = \frac{1}{2} \binom{2n+2}{n+1} = \frac{1}{2} \binom{n+1+n+1}{n+1},$$

that is,

$$\begin{aligned}
W(n+1) &= W(n) + \frac{1}{2} \sum_{k=0}^{n+1} \left[\binom{n+1+k}{k} \frac{1}{2^k} \right] \\
&= W(n) + \frac{1}{2} W(n+1),
\end{aligned}$$

that is,

$$W(n+1) = 2W(n) = 2^{n+1}.$$