CS21001 Discrete Structures

Tutorial 7

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1. Consider a linear recurrence relation with constant coefficients having characteristic equation $(x-r)^\mu$ for some $\mu \in \mathbb{N}$, and with a non-homogeneous part $f(n) = n^t r^n$. Using the theory of generating functions, prove that the particular solution for this recurrence relation is of the form

$$n^\mu (u_t n^t + u_{t-1} n^{t-1} + \cdot + u_2 t^2 + u_1 t + u_0) r^n.$$

2. Foosia and Barland play a long series of ODI matches. In the first match, Foosia bats first. After that, the team that wins a match must bat first in the next match. For each team, the probability of win is p if it bats first. Assume that $0 < p < 1$. Find the probability p_n that Foosia wins the n-th match. What is $\lim_{n\to\infty} p_n$?

3. Pell numbers are defined as $P_0 = 0$, $P_1 = 2$, $P_n = 2P_{n-1} + P_{n-2}$ for $n \geqslant 2$.

- **(a)** Deduce a closed-form formula for P_n.
- **(b)** Prove that $\begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n$ for all $n \geqslant 1$.
- **(c)** Prove that $\lim_{n\to\infty} \dfrac{P_{n-1} + P_n}{P_n} = \sqrt{2}$.
- **(d)** Prove that if P_n is prime, then n is also prime.

4. The Pell–Lucas numbers are defined as $Q_0 = Q_1 = 2$, $Q_n = 2Q_{n-1} + Q_{n-2}$ for $n \geqslant 2$.

- **(a)** Deduce a closed-form formula for Q_n.
- **(b)** Prove that $Q_n = P_{2n}/P_n$ for all $n \geqslant 1$.

5. Let $a_0 = 1$, and $a_n = \frac{5}{2} a_{n-1} - a_{n-2}$ for all $n \geqslant 2$. Find a_1 such that the sequence a_n converges.

6. A set of natural numbers is called *selfish* if it contains its size as a member. Let s_n denote the number of selfish subsets of $\{1, 2, 3, \ldots, n\}$ for $n \geqslant 1$. Develop a recurrence relation for s_n, and solve it.

7. Let us call a selfish set A minimal if no proper subset of A is selfish. Let S_n denote the number of minimal selfish subsets of $\{1, 2, 3, \ldots, n\}$. Develop a recurrence relation for S_n, and solve it.

8. The triangular numbers are defined as $t_n = 1 + 2 + 3 + \cdots + n = n(n+1)/2$ for $n \geqslant 0$. Define $a_n = \sum_{i=0}^{n} t_i$ for $n \geqslant 0$. Find a recurrence relation for a_n, and solve it.

9. Let a_n, $n \geqslant 0$, be the count of strings over $\{0, 1, 2\}$ containing no consecutive 1's and no consecutive 2's. Find a recurrence relation for a_n, and solve it.

10. Let a_n, $n \geqslant 0$, denote the number of binary strings of length n, not containing the pattern 101. Develop a recurrence relation for a_n, and solve it.

11. Let D_n, $n \geqslant 1$, denote that number of derangements (permutations without fixed points) of $1, 2, 3, \ldots, n$.

- **(a)** Prove that $D_n = (n-1)(D_{n-1} + D_{n-2})$ for all $n \geqslant 3$.
- **(b)** Deduce that $D_n = nD_{n-1} + (-1)^n$ for all $n \geqslant 3$.
- **(c)** Solve for D_n.

12. Let a_n, $n \geqslant 1$, satisfy $a_1 = 1$, and $a_n = \begin{cases} 2a_{n-1} & \text{if } n \text{ is odd} \\ 2a_{n-1} + 1 & \text{if } n \text{ is even} \end{cases}$ for $n \geqslant 2$. Develop a recurrence relation for a_n, that holds for both odd and even n, and solve it.

13. Solve the recurrence relation: $a_0 = 1$, $a_1 = 2$, $a_n = a_{n-1} + 2a_{n-2} + n^2 + 2^n$ for $n \geqslant 2$.

14. Solve the recurrence relation: $a_0 = 1$, $a_1 = 2$, $a_n = 4a_{n-2} + 2^n + n3^n$ for $n \geqslant 2$.

15. Solve the recurrence relation: $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, $a_n = a_{n-1} + a_{n-2} - a_{n-3} + n^2 + n + (-1)^n$ for $n \geqslant 3$.

16. Solve the recurrence relation: $a_0 = 1$, $a_1 = 2$, $a_2 = 4$, and $a_n = \dfrac{2a_{n-1}^3}{a_{n-3}^2}$ for $n \geqslant 3$.

17. Solve the recurrence relation $na_n = (n+1)a_{n-1} + 2n$ for $n \geqslant 1$, with the initial condition $a_0 = 0$.

18. Solve the recurrence relation $a_n = na_{n-1} + n(n-1)a_{n-2} + n!$ for $n \geqslant 2$, with $a_0 = 0$, $a_1 = 1$.

19. Solve the recurrence relation $a_0 = \frac{2}{3}$, and $a_n = 2a_{n-1}^2 - 1$ for $n \geqslant 1$.

20. A sequence a_n is defined recursively as $a_1 = 1$, $a_2 = 2$, $a_3 = 24$, and $a_n = \dfrac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}$ for $n \geqslant 4$. Prove that a_n is an integer for all $n \in \mathbb{N}$.

21. Consider the recurrence relation $a_n = a_{n-1} + 3a_{n-2} - a_{n-3}$ for $n \geqslant 3$. Find a matrix A such that $\begin{pmatrix} a_3 \\ a_4 \\ a_5 \end{pmatrix} = A \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$. Express $\begin{pmatrix} a_{3n} \\ a_{3n+1} \\ a_{3n+2} \end{pmatrix}$ in terms of A and a_0, a_1, a_2, for all $n \in \mathbb{N}_0$.

22. So far, we have solved recurrence relations. In this exercise, we reverse this process, that is, from a sequence, we generate a recurrence relation, of which the given sequence is a solution. We concentrate only on linear recurrence relations with constant coefficients (homogeneous/non-homogeneous). Solve the following parts for the given sequences with the orders of the recurrence relations as specified.

(a) $(2+\sqrt{3})^n + (2-\sqrt{3})^n$, order two.
(b) $2^n + 3^n$, order two.
(c) $2^n + 3^n$, order one.
(d) $2^n + n3^n$, order two.
(e) $2^n + n3^n$, order one.
(f) $2^n + n3^n + n^2 4^n$, order three.
(g) $2^n + n3^n + n^2 4^n$, order two.
(h) $2^n + n3^n + n^2 4^n$, order one.

23. Let $A(x) = 1 + \dfrac{1}{\sqrt{1-2x}}$ be the (ordinary) generating function of a sequence a_n, $n \geqslant 0$. Develop a recurrence relation for the sequence.

24. Let a_n denote the number of strings w of length n over the alphabet $\{A, C, G, T\}$ such that the number of T in w is a multiple of 3. Find a closed-form expression for a_n.

25. How many lines are printed by the call $f(n)$ for an integer $n \geqslant 0$?

```
void f ( int n )
{
    int m;
    printf("Hi\n");
    m = n - 1;
    while (m >= 0) { f(m); m -= 2; }
}
```

26. How many lines are printed by the call $g(n, 0, 0)$ for an integer $n \geqslant 0$?

```
void g ( int n, int i, int flag )
{
    if (i == n) { printf("Hola\n"); return; }
    g(n, i+1, flag); g(n, i+1, flag); g(n, i+1, flag);
    if (flag == 0) g(n, i+1, 1);
}
```

27. **(a)** How many strings of length n over the alphabet $\{A, C, G, T\}$ are there, in which T never appears after A? Notice that there is no restriction on the appearances of T before the first occurrence of A.

(b) Modify the function `g()` of the last exercise so as to print precisely the strings of Part (a).

28. How many strings of length n over the alphabet $\{A, C, G, T\}$ are there, in which the pattern TT (two consecutive T's) never appears after A? Note that TT may appear before the first occurrence of A, and that single isolated T's may appear after A.