CS21001 Discrete Structures
Tutorial 6

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1. Find the generating functions of the following sequences.

(a) $1,0,0,1,0,0,1,0,0,\ldots$
(b) $1,1,0,1,1,0,1,1,0,\ldots$
(c) $1,3,5,7,9,11,13,\ldots$
(d) $2,4,8,14,22,32,44,\ldots$
(e) $1,0,0,2,0,0,3,0,0,4,0,0,\ldots$
(f) $1,2,0,3,4,0,5,6,0,7,8,0,\ldots$
(g) $1/1,1/2,1/3,1/4,1/5,\ldots$
(h) $H_0,H_1,H_2,H_3,\ldots$ (where H_n is the n-th harmonic number)

2. Let $A(x)$ be the generating function for the sequence $a_0,a_1,a_2,\ldots$. Express the generating functions of the following sequences in terms of $A(x)$.

(a) $a_0,2a_1,3a_2,4a_3,5a_4,\ldots$
(b) $a_0,a_1-a_0,a_2-a_1,a_3-a_2,\ldots$
(c) $a_0,a_0-a_1,a_0-a_1+a_2,a_0-a_1+a_2-a_3,\ldots$
(d) $a_0,a_1,a_2-a_0,a_3-a_1,a_4-a_2+a_0,a_5-a_3+a_1,a_6-a_4+a_2-a_0,a_7-a_5+a_3-a_1,\ldots$
(e) $a_0+a_1,a_1+a_2,a_2+a_3,a_3+a_4,\ldots$
(f) $a_0+a_1,a_2+a_3,a_4+a_5,a_6+a_7,\ldots$

3. Find the number of solutions of $x_1+x_2+x_3+x_4=n$ with integer-valued variables satisfying $x_1 \geqslant -1$, $x_2 \geqslant -2$, $x_3 \geqslant 3$, and $x_4 \geqslant 4$.

4. How many bit strings of length n are there in which 1's always occur in contiguous pairs? You should consider strings of the form 0011011110, but not of the form 0110111110, because the last 1 is not paired.

5. Use generating functions to prove that every positive integer has a unique binary representation (without leading zero bits).

6. **(a)** For $n \in \mathbb{N}$, denote by $\sigma(n)$ the sum of all positive integral divisors of n. We also take $\sigma(0)=0$. Find the generating function of the sequence $\sigma(0),\sigma(1),\sigma(2),\ldots,\sigma(n),\ldots$.

(b) If $u,v \in \mathbb{N}$ are coprime, prove that $\sigma(uv)=\sigma(u)\sigma(v)$. Hence deduce a closed-form expression for $\sigma(n)$ with n having the prime factorization $n=p_1^{e_1}p_2^{e_2}\cdots p_t^{e_t}$.

7. Let F_n, $n \geqslant 0$, denote the Fibonacci sequence. Prove that $\displaystyle\sum_{n\in\mathbb{N}_0}\frac{F_n}{2^n}=2$.

8. Let $A(x)$ be the generating function of the sequence $a_0,a_1,a_2,a_3,\ldots$ of real numbers. Prove that $1/A(x)$ is the generating function of a sequence if and only if $a_0 \neq 0$.

9. **(a)** Find the probability generating function of the binomial distribution $\Pr[B_{n,p}=r]=\binom{n}{r}p^r(1-p)^{n-r}$ for $r=0,1,2,\ldots,n$. Hence deduce the expectation $\mathrm{E}[B_{n,p}]$.

(b) Find the probability generating function of the uniform distribution $\Pr[U_{a,b}=r]=\dfrac{1}{b-a+1}$ for $r=a,a+1,a+2,\ldots,b$ (where $a,b\in\mathbb{Z}$ with $a\leqslant b$). Hence deduce the expectation $\mathrm{E}[U_{a,b}]$.

10. The generating function $A(x)$ of a sequence $a_0,a_1,a_2,a_3,\ldots$ satisfies $A'(x)=1+A(x)$. Prove that $A(x)=(a_0+1)e^x-1$.

11. Let $A(x)$ be the exponential generating function of a sequence $a_0,a_1,a_2,\ldots$. Express the exponential generating functions of the following sequences in terms of $A(x)$.

(a) $a_0,2a_1,3a_2,4a_3,5a_4,\ldots$
(b) $0,a_0,a_1,a_2,a_3,a_4,\ldots$
(c) $a_1,a_2,a_3,a_4,a_5,\ldots$
(d) $a_0,a_1-a_0,a_2-a_1,a_3-a_2,\ldots$

12. Let $A(x)$ and $B(x)$ be the exponential generating functions of the two sequences $a_0, a_1, a_2, a_3, \ldots$ and $b_0, b_1, b_2, b_3, \ldots$. Of what sequence is $A(x)B(x)$ the EGF?

13. Prove the following identities involving the Stirling numbers $S(n,k)$ of the second kind. Take $S(0,0) = 1$, and $S(n,k) = 0$ for $n < k$.

(a) $\displaystyle\sum_{n \in \mathbb{N}_0} S(n,k)x^n = \prod_{r=1}^{k} \frac{x}{1-rx}$.

(b) $\displaystyle\sum_{n \in \mathbb{N}_0} S(n,k)\frac{x^n}{n!} = \frac{(e^x-1)^k}{k!}$.

14. Deduce that the exponential generating function of the Bell numbers B_n, $n \in \mathbb{N}_0$, is e^{e^x-1}.

15. Let A be a (real-valued) random variable, and $n \in \mathbb{N}_0$. The n-th moment of A (about zero) is defined as $\mu_n = \mathrm{E}[A^n]$. The exponential generating function of the sequence $\mu_0, \mu_1, \mu_2, \mu_3, \ldots$ is called the *moment generating function* $M_A(x)$ of A. Prove that $M_A(x) = \mathrm{E}[e^{xA}]$ (provided that this expectation exists).

16. Find the moment generating functions of the following random variables.

(a) Binomial distribution: $\Pr[B_{n,p} = r] = \binom{n}{r} p^r (1-p)^r$ for $r = 0, 1, 2, \ldots, n$.

(b) Geometric distribution: $\Pr[G_p = r] = (1-p)^{r-1}p$ for $r \in \mathbb{N}$.

(c) The discrete uniform distribution: $\Pr[U_{a,b} = r] = \dfrac{1}{b-a+1}$ for $r = a, a+1, a+2, \ldots, b$.

(d) The continuous uniform distribution: $\Pr[C_{a,b} = r] = \dfrac{1}{b-a}$ for $r \in [a,b]$.

(e) Poisson distribution: $\Pr[P_\lambda = r] = e^{-\lambda}\dfrac{\lambda^r}{r!}$ for $r \in \mathbb{N}_0$.

17. Generating functions with multiple variables are sometimes used. Suppose that r elements are chosen from $\{1, 2, 3, \ldots, n\}$ such that the sum of the chosen elements is s. We want to count how many such collections are possible. Argue that this count is the coefficient of $x^r y^s$ of $(1+xy)(1+xy^2)(1+xy^3)\cdots(1+xy^n)$. Find the two-variable generating function of these counts if the r elements are chosen from $\{1, 2, 3, \ldots, n\}$ with repetitions allowed.

18. Let a_n, $n \geqslant 0$, be the sequence satisfying

$$\begin{aligned} a_0 &= 1, \\ a_n &= 2 + 2a_0 + 2a_1 + 2a_2 + \cdots + 2a_{n-2} + a_{n-1} \text{ for } n \geqslant 1. \end{aligned}$$

Deduce that the generating function of this sequence is $\dfrac{1+x}{1-2x-x^2}$. Solve for a_n.

19. Let a_n, $n \geqslant 0$, be the sequence satisfying

$$\begin{aligned} a_0 &= 0, \\ a_1 &= 1, \\ a_n &= a_{n-1} + \sum_{k=1}^{n-2} a_k a_{n-1-k} \text{ for } n \geqslant 2. \end{aligned}$$

Prove that the generating function for this sequence is $\dfrac{1-x-\sqrt{1-2x-3x^2}}{2x}$. Solve for a_n.

20. Prove that the n-th Catalan number $C(n)$ is equal to the following.

(a) The number of preorder listings of binary search trees storing only the keys $1, 2, 3, \ldots, n$.

(b) The number of ways of drawing n non-intersecting chords with $2n$ distinct endpoints placed on the circumference of a circle.

(c) The number of ways a convex $(n+2)$-gon can be cut into triangles.

21. Solve the following recurrence relation using generating functions: $a_0 = 1$, $a_1 = 2$, $a_2 = 3$, $a_n = 4a_{n-1} - 5a_{n-2} + 2a_{n-3} + 1$ for $n \geqslant 3$.