CS21001 Discrete Structures
Tutorial 5

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1. Let A, B be sets. Prove or disprove:

(a) If A is countable and $A \subseteq B$, then B is countable.
(b) If A is uncountable and $A \subseteq B$, then B is uncountable.
(c) If A and B are countable, then $A \cap B$ is countable.
(d) If A and B are uncountable, then $A \cap B$ is uncountable.

2. **(a)** Prove that the set of all finite subsets of $\mathbb{N}$ is countable.
(b) Conclude that the set of all infinite subsets of $\mathbb{N}$ is uncountable.

3. Let A be a finite set.

(a) Prove that the set of all functions $A \to \mathbb{N}$ is countable.
(b) Let $|A| \geqslant 2$. Prove that the set of all functions $\mathbb{N} \to A$ is uncountable.
(c) Let $|A| \geqslant 2$. Prove that the set of all functions $\mathbb{N} \to A$ is equinumerous with $\mathbb{R}$.

4. **(a)** Let $\mathbb{Z}[X]$ denote the set of polynomials in one variable X and with integer coefficients. Prove that $\mathbb{Z}[X]$ is countable.
(b) Let k be a fixed positive integer. Prove that the set $\mathbb{Z}[X_1, X_2, \ldots, X_k]$ of multivariate polynomials with integer coefficients is countable.
(c) Prove that the set $\mathbb{Z}[X_1, X_2, \ldots, X_k, \ldots]$ of polynomials with countably infinite variables and with integer coefficients is countable.

Note: A polynomial contains, by definition, only finitely many non-zero terms. Moreover, each term is required to contain only finitely many variables with non-zero exponents.

5. A real or complex number α is called *algebraic* if $f(\alpha) = 0$ for some non-zero univariate polynomial $f(X)$ with integer coefficients. Let $\mathbb{A}$ denote the set of all algebraic numbers. We have $\mathbb{A} \subseteq \mathbb{C}$.

(a) Prove that $\mathbb{A}$ is countable.
(b) Conclude that there are uncountably many transcendental numbers.

6. Let $f(X,Y)$ be a non-zero bivariate polynomial with integer coefficients. Let $\mathbb{V}$ denote the set of all pairs $(\alpha, \beta) \in \mathbb{C}^2$ such that $f(\alpha, \beta) = 0$. Prove or disprove: $\mathbb{V}$ is countable.

7. Provide explicit bijections between the following pairs of sets.

(a) The sets $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$.
(b) The set of rational numbers in the real interval $[0,1)$ and the set $\mathbb{Q}$ of all rational numbers.
(c) The set of irrational numbers in the real interval $[0,1)$ and the set of all irrational numbers.
(d) The real interval $[0,1)$ and $\mathbb{R}$.
(e) The real interval $(0,1)$ and $\mathbb{R}$.
(f) The real intervals $[0,1)$ and $[a,b)$ for any $a, b \in \mathbb{R}$, $a < b$.
(g) The real intervals $[0,1)$ and $(0,1)$.
(h) The real intervals $[0,1]$ and $(0,1)$.

8. Let A, B be sets, where A is equinumerous with $\mathbb{R}$ and B is equinumerous with $\mathbb{N}$.

(a) Prove that $A \cup B$ is equinumerous with $\mathbb{R}$.
(b) Prove that the Cartesian product $A \times B$ is equinumerous with $\mathbb{R}$.

9. Prove that the set $\{a + \mathrm{i}b \mid a, b \in \mathbb{Z}\}$ of Gaussian integers is countable.

10. Prove that the real interval $[0,1)$ is equinumerous with the unit square $S = \{x + \mathrm{i}y \mid x, y \in [0,1)\}$ in the complex plane.

11. Provide a diagonalization argument to prove that the set A of all infinite bit sequences is uncountable.

12. **(a)** Prove that the set $\mathbb{Q}[[X]]$ of all power series with rational coefficients is uncountable.

(b) Prove that the set $\mathbb{Q}(X) = \{f(X)/g(X) \mid g(X) \neq 0\}$ of all rational functions with rational coefficients is countable.

(c) Conclude that $\mathbb{Q}[[X]]$ contains a power series which is not the power series expansion of any rational function in $\mathbb{Q}(X)$. Can you identify any such power series explicitly?

13. Prove that the union of two sets each equinumerous with $\mathbb{R}$ is again equinumerous with $\mathbb{R}$

14. Prove that the set of all permutations of $\mathbb{N}$ is not countable.

15. Let A be a set of size $\geqslant 2$ (A may be infinite). Modify the diagonalization proof to establish that there cannot exist a bijection between A and the set of all *non-empty* subsets of A.

16. Let $S = (s_1, s_2, s_3, \ldots)$ and $T = (t_1, t_2, t_3, \ldots)$ be two infinite bit sequences. We say that S and T have the *same tail* if there exists $N \in \mathbb{N}$ such that $s_n = t_n$ for all $n \geqslant N$. Prove that for any given sequence S, the set of sequences having the same tail as S is countable.

17. Consider the relation ρ on $\mathbb{R}$ as $a\ \rho\ b$ if and only if $a - b \in \mathbb{Q}$.

(a) Prove that ρ is an equivalent relation.
(b) Prove that each equivalence class of ρ is countably infinite.
(c) Conclude that the set A/ρ of equivalence classes is uncountable.

18. A subset $S \subseteq \mathbb{R}$ is called *bounded* if S has both a lower bound and an upper bound in $\mathbb{R}$. Which of the following sets is/are countable/uncountable? Why?

(a) The set of all bounded subsets of $\mathbb{Z}$.
(b) The set of all bounded subsets of $\mathbb{Q}$.
(c) The set of all bounded subsets of $\mathbb{R}$.

19. [*Cantor set*] Start with the real interval $I = [0,1]$. Remove the open middle one-third $(\frac{1}{3}, \frac{2}{3})$ from $[0,1]$. This leaves us with two closed intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. Remove the open middle one-thirds of these two intervals, that is, $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$. The portion of I that remains now consists of the four closed intervals $[0, \frac{1}{9}]$, $[\frac{2}{9}, \frac{1}{3}]$, $[\frac{2}{3}, \frac{7}{9}]$, and $[\frac{8}{9}, 1]$. Again, remove the open middle one-thirds of these four intervals, leaving eight closed subintervals in I. Repeat this process infinitely often. Let C be the subset of I that remains after this infinite process. Prove that C is uncountable.

Note: The cantor set C is one of the first explicitly constructed examples of *fractal sets*.

20. Repeat Cantor's process of the last exercise with the exception that you remove the closed middle one-thirds of the remaining intervals. That is, in the first step, you remove $[\frac{1}{3}, \frac{2}{3}]$, in the second step, you remove $[\frac{1}{9}, \frac{2}{9}]$ and $[\frac{7}{9}, \frac{8}{9}]$, and so on. Now, let D be the subset of $I = [0,1]$, that remains after this infinite process. Evidently, D is a proper subset of C. Is D uncountable too?