CS21001 Discrete Structures
Tutorial 4

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1. Let A be the set of all non-empty finite subsets of $\mathbb{Z}$. Define a relation ρ on A as: $U \ \rho \ V$ if and only if $\min(U) = \min(V)$. Also define the relation σ on A as: $U \ \sigma \ V$ if and only if $\min(U) \leqslant \min(V)$. Finally, define a relation τ on A as: $U \ \tau \ V$ if and only if either $U = V$ or $\min(U) < \min(V)$.

(a) Prove that ρ is an equivalence relation on A.
(b) Identify good representatives from the equivalence classes of ρ.
(c) Define a bijection between the quotient set A/ρ and $\mathbb{Z}$.
(d) Prove or disprove: σ is a partial order on A.
(e) Prove or disprove: τ is a partial order on A.

2. Let $f : A \to B$ be a function and σ an equivalence relation on B. Define a relation ρ on A as: $a \ \rho \ a'$ if and only if $f(a) \ \sigma \ f(a')$.

(a) Prove that ρ is an equivalence relation on A.
(b) Define a map $\bar{f} : A/\rho \to B/\sigma$ as $[a]_\rho \mapsto [f(a)]_\sigma$. Prove that f is well-defined.
(c) Prove that $\bar{f}$ is injective.
(d) Prove or disprove: If f is a bijection, then so also is $\bar{f}$.
(e) Prove or disprove: If $\bar{f}$ is a bijection, then so also is f.

3. Let $f : A \to B$ be a function, ρ an equivalence relation on A, and σ an equivalence relation on B. Suppose further that if $a \ \rho \ a'$, then $f(a) \ \sigma \ f(a')$. Define the map $\bar{f} : A/\rho \to B/\sigma$ as $\bar{f}([a]_\rho) = [f(a)]_\sigma$. Argue that $\bar{f}$ is well-defined. Prove or disprove the following statements.

(a) $\bar{f}$ is injective.
(b) If f is a bijection, then so also is $\bar{f}$.
(c) If $\bar{f}$ is a bijection, then so also is f.

4. Let $f : A \to B$ be a function, ρ an equivalence relation on A, and σ an equivalence relation on B. Suppose further that if $f(a) \ \sigma \ f(a')$, then $a \ \rho \ a'$. Show by an explicit example that the association $\bar{f} : A/\rho \to B/\sigma$ given by $\bar{f}([a]_\rho) = [f(a)]_\sigma$ is not necessarily a function.

5. Define the relation ρ on $\mathbb{R}$ as: $a \ \rho \ b$ if and only if $a - b \in \mathbb{Z}$.

(a) Prove that ρ is an equivalence relation on $\mathbb{R}$.
(b) Find good representatives for the equivalence classes of ρ.
(c) Provide an explicit bijection between $\mathbb{R}/\rho$ and the real interval $(3, 5]$.

6. Let m, n be positive integers. Prove that the assignment $f : \mathbb{Z}_m \to \mathbb{Z}_n$ taking $[a]_m \mapsto [a]_n$ is well-defined if and only if m is an integral multiple of n.

7. [*Genesis of rational numbers*] Define a relation ρ on $A = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ as $(a, b) \ \rho \ (c, d)$ if and only if $ad = bc$. Prove that ρ is an equivalence relation. Argue that A/ρ is essentially the set $\mathbb{Q}$ of rational numbers. In abstract algebra, we say that $\mathbb{Q}$ is the *field of fractions* of the integral domain $\mathbb{Z}$. The equivalence class $[(a, b)]$ is conventionally denoted by $\frac{a}{b}$.

8. [*Genesis of real numbers*] An infinite sequence $a_1, a_2, a_3, \ldots$ of rational numbers is called a *Cauchy sequence* if given any real $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|a_m - a_n| < \varepsilon$ for all $m, n \geqslant N$. Let C denote the set of all Cauchy sequences of rational numbers.

(a) Prove that any Cauchy sequence converges, that is, has a limit.
(b) Establish that the limit of a Cauchy sequence may be irrational.
(c) Define a relation ρ on C as $S \ \rho \ T$ if and only if $\lim S = \lim T$. Prove that ρ is an equivalence relation.
(d) Convince yourself that C/ρ is essentially the set $\mathbb{R}$ of real numbers. This process of the generation of $\mathbb{R}$ from $\mathbb{Q}$ is called *completion*. Another method of defining $\mathbb{R}$ uses *Dedekind cuts*.

9. Give an example of a poset A and a non-empty subset S of A such that S has lower bounds in A, but $\mathrm{glb}(S)$ does not exist.

10. Let ρ be a total order on A. We call ρ a *well-ordering* of A if every non-empty subset of A contains a least element. Which of the following sets is/are well-ordered under the standard $\leqslant$ relation: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}^+$, $\mathbb{R}$?

11. In this exercise, we plan to construct a well-ordering of $A = \mathbb{N} \times \mathbb{N}$.

(a) Define a relation ρ on A as $(a,b)\ \rho\ (c,d)$ if and only if $a \leqslant c$ or $b \leqslant d$. Prove or disprove: ρ is a well-ordering of A.
(b) Define a relation σ on A as $(a,b)\ \sigma\ (c,d)$ if and only if $a \leqslant c$ and $b \leqslant d$. Prove or disprove: σ is a well-ordering of A.
(c) Define a relation $\leqslant_L$ on A as $(a,b) \leqslant_L (c,d)$ if either (i) $a < c$ or (ii) $a = c$ and $b \leqslant d$. Prove that $\leqslant_L$ is a well-ordering of A.
(d) Prove or disprove: An infinite subset of A may contain a maximum element with respect to $\leqslant_L$.

12. A *string* is a finite ordered sequence of symbols from a finite alphabet. We start with a predetermined total ordering of the alphabet, and then define the usual dictionary order on strings. Prove that this dictionary order (called *lexicographic ordering*) is a total ordering. Is it also a well-ordering?

13. Define a relation $\leqslant_{DL}$ on $A = \mathbb{N} \times \mathbb{N}$ as follows. Take $(a,b),(c,d) \in A$ and call $(a,b) \leqslant_{DL} (c,d)$ if either (i) $a+b < c+d$, or (ii) $a+b = c+d$ and $a \leqslant c$.

(a) Prove that $\leqslant_{DL}$ is a partial order on A.
(b) Prove that $\leqslant_{DL}$ is a total order on A.
(c) Is A well-ordered by $\leqslant_{DL}$?
(d) Prove or disprove: An infinite subset of A may contain a maximum element.

Note: The ordering $\leqslant_{DL}$ on A is called the *degree-lexicographic ordering*. Identify $(a,b) \in A$ with the monomial X^aY^b. First, order monomials with respect to their degrees. For two monomials of the same degree, apply lexicographic ordering. For example, $XY^3 \leqslant_{DL} Y^5$ and $XY^3 \leqslant_{DL} X^2Y^2$.

14. Generalize the degree-lexicographic ordering on $\mathbb{N}^n$ for any fixed $n \geqslant 3$.

15. Consider the following relation ρ on the set $\mathbb{Q}^+$ of all positive rational numbers. Take $a/b, c/d \in \mathbb{Q}^+$ with $\gcd(a,b) = \gcd(c,d) = 1$. Call $(a/b)\,\rho\,(c/d)$ if and only if either (i) $a+b < c+d$ or (ii) $a+b = c+d$ and $a \leqslant c$. Prove that ρ is a total order. Prove that $\mathbb{Q}^+$ is well-ordered by ρ.

16. Construct a well-ordering of $\mathbb{Q}$.

17. Let A be the set of all functions $\mathbb{N} \to \mathbb{N}$. For $f,g \in A$, define $f \leqslant g$ if and only if $f(n) \leqslant g(n)$ for all $n \in \mathbb{N}$. prove that $\leqslant$ is a partial order on A. Is $\leqslant$ also a total order?

18. Let A be the set of all functions $\mathbb{N}_0 \to \mathbb{R}^+$.

(a) Define a relation Θ on A as $f\ \Theta\ g$ if and only if $f = \Theta(g)$. Prove that Θ is an equivalence relation.
(b) Define a relation O on A as $f\ \mathrm{O}\ g$ if and only if $f = \mathrm{O}(g)$. Argue that O is not a partial order.

Define a relation O on A/Θ as $[f]\ \mathrm{O}\ [g]$ if and only if $f = \mathrm{O}(g)$.

(c) Establish that the relation O is well-defined.
(d) Prove that O is a partial order on A/Θ.
(e) Prove or disprove: O is a total order on A/Θ.
(f) Prove or disprove: A/Θ is a lattice under O.

19. Let k be a fixed positive integer. Define a relation $\leqslant$ on $A = \mathbb{Z}^k$ as: $(a_1,a_2,\ldots,a_k) \leqslant (b_1,b_2,\ldots,b_k)$ if and only if $a_i \leqslant b_i$ for all $i = 1,2,\ldots,k$. Prove that A is a lattice under this relation.

20. Let A be a poset under the relation ρ. Prove or disprove:

(a) If ρ is a total order, then A is a lattice.
(b) If A is a lattice, then ρ is a total order.

21. Let A be a poset. We call A a *meet-semilattice* (resp. *join-semilattice*) if $\mathrm{glb}(a,b)$ (resp. $\mathrm{lub}(a,b)$) exists for all $a,b \in A$. A is a lattice if and only if it is both a meet-semilattice and a join-semilattice. Give examples of:

(a) A meet-semilattice which is not a lattice.
(b) A join-semilattice which is not a lattice.