CS21001 Discrete Structures

Tutorial 3

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1. Let $S(m,n)$ be the Stirling numbers of the second kind. Let us define *falling factorials* as

$$x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1).$$

Prove the identity $x^m = \sum_{n=0}^{m} S(m,n)x^{\underline{n}}$.

2. Prove the identity $S(m,n) = \sum_{r=n-1}^{m} \binom{m}{r} S(r,n-1)$.

3. The n-th *Bell number* is defined as the total number of partitions of an n-set (into any number of parts), so

$$B_n = \sum_{k=0}^{n} S(n,k).$$

Prove the identity $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$.

4. Let $s(n,m)$ denote the number of permutations of $1,2,3,\ldots,n$, that have exactly m cycles. For example, the permutation $3,1,6,8,2,5,7,4$ (for $n=8$) has three cycles $(1,3,6,5,2),(4,8),(7)$. The numbers $s(n,m)$ are called *Stirling numbers of the first kind*. Prove that $s(m,n) = s(m-1,n-1)+(m-1)s(m-1,n)$.

5. Define the *rising factorials* as

$$x^{\overline{m}} = x(x+1)(x+2)\cdots(x+m-1).$$

Prove the identity $x^{\overline{m}} = \sum_{n=0}^{m} s(m,n)x^n$. How can you express the falling factorial $x^{\underline{m}}$ in terms of the Stirling numbers of the first kind?

6. Let $f: A \to B$ be a function. Prove the following assertions.

- **(a)** $S \subseteq f^{-1}(f(S))$ for every $S \subseteq A$. Give an example where the inclusion is proper.
- **(b)** f is injective if and only if $S = f^{-1}(f(S))$ for every $S \subseteq A$.
- **(c)** $f(f^{-1}(T)) \subseteq T$ for every $T \subseteq B$. Give an example where the inclusion is proper.
- **(d)** f is surjective if and only if $f(f^{-1}(T)) = T$ for every $T \subseteq B$.
- **(e)** $f(f^{-1}(f(S))) = f(S)$ for all $S \subseteq A$.
- **(f)** $f^{-1}(f(f^{-1}(T))) = f^{-1}(T)$ for all $T \subseteq B$.

7. Let $f: A \to B$ and $g: B \to C$ be functions.

- **(a)** Prove that if the function $g \circ f: A \to C$ is injective, then f is injective.
- **(b)** Give an example in which $g \circ f$ is injective, but g is not injective.
- **(c)** Prove that if $g \circ f$ is surjective, then g is surjective.
- **(d)** Give an example in which $g \circ f$ is surjective, but f is not surjective.

8. A function $f: \mathbb{Z} \to \mathbb{Z}$ is called *nilpotent* if for some $n \in \mathbb{N}$ we have $f^n(a) = 0$ for all $a \in \mathbb{Z}$.

- **(a)** Give an example of a non-constant nilpotent function.
- **(b)** Prove or disprove: The function $f(a) = \lfloor |a|/2 \rfloor$ is nilpotent.

9. For a function $f: A \to B$, define a function $\mathscr{F}: \mathscr{P}(A) \to \mathscr{P}(B)$ as $\mathscr{F}(S) = f(S)$ for all $S \subseteq A$. Prove that:

- **(a)** $\mathscr{F}$ is injective if and only if f is injective.
- **(b)** $\mathscr{F}$ is surjective if and only if f is surjective.
- **(c)** $\mathscr{F}$ is bijective if and only if f is bijective.

10. A function $f:\mathbb{R}\to\mathbb{R}$ is called *monotonic increasing* if $f(a)\leqslant f(b)$ whenever $a\leqslant b$. It is called *strictly monotonic increasing* if $f(a)<f(b)$ whenever $a<b$. One can define *monotonic decreasing* and *strictly monotonic decreasing* functions in analogous ways.

(a) Prove that a strictly monotonic increasing function is injective.
(b) Demonstrate that an injective function $\mathbb{R}\to\mathbb{R}$ need not be strictly increasing or strictly decreasing.
(c) Prove that a continuous injective function $\mathbb{R}\to\mathbb{R}$ is either strictly increasing or strictly decreasing.

11. Let $x\in\mathbb{R}$ and $m,n\in\mathbb{N}$. Prove the following assertions about the floor and ceiling functions.

(a) $\left\lceil \frac{x}{n} \right\rceil = \left\lfloor \frac{x+n-1}{n} \right\rfloor$.

(b) $\left\lfloor \left\lfloor \frac{x}{m} \right\rfloor /n \right\rfloor = \left\lfloor \frac{x}{mn} \right\rfloor$ and $\left\lceil \left\lceil \frac{x}{m} \right\rceil /n \right\rceil = \left\lceil \frac{x}{mn} \right\rceil$.

(c) $\lfloor mx \rfloor = \lfloor x \rfloor + \left\lfloor x+\frac{1}{m} \right\rfloor + \left\lfloor x+\frac{2}{m} \right\rfloor + \cdots + \left\lfloor x+\frac{m-1}{m} \right\rfloor$.

12. Let A be the set of all non-empty finite subsets of $\mathbb{Z}$. Define a relation τ on A as: $U\,\tau\,V$ if and only if either $U=V$ or $\min(U)<\min(V)$. Prove or disprove: τ is a partial order on A.

13. Let A be the set of all functions $\mathbb{R}\to\mathbb{R}$. Define relations ρ,σ,τ on A as follows.

$$\begin{aligned} f\,\rho\,g &\quad \text{if and only if} \quad f(a)\leqslant g(a) \text{ for all } a\in\mathbb{R},\\ f\,\sigma\,g &\quad \text{if and only if} \quad f(0)\leqslant g(0),\\ f\,\tau\,g &\quad \text{if and only if} \quad f(0)=g(0). \end{aligned}$$

Argue which of the relations ρ,σ,τ is/are equivalence relation(s). Argue which is/are partial order(s).

14. Let ρ be a relation on a set A. Define $\rho^{-1}=\{(b,a)\mid(a,b)\in\rho\}$. Also for two relations ρ,σ on A, define the composite relation $\rho\circ\sigma$ as $(a,c)\in\rho\circ\sigma$ if and only if there exists $b\in A$ such that $(a,b)\in\rho$ and $(b,c)\in\sigma$. Prove the following assertions.

(a) ρ is both symmetric and antisymmetric if and only if $\rho\subseteq\{(a,a)\mid a\in A\}$.
(b) ρ is transitive if and only if $\rho\circ\rho=\rho$.
(c) If ρ is non-empty, then ρ is an equivalence relation if and only if $\rho^{-1}\circ\rho=\rho$.
(d) ρ is a partial order if and only if ρ^{-1} is a partial order.

15. A repunit is an integer of the form $111\ldots1$. Prove that any $n\in\mathbb{N}$ with $\gcd(n,10)=1$ divides a repunit.

16. You pick six points in a 3×4 rectangle. Prove that two of these points must be at a distance $\leqslant\sqrt{5}$.

17. You pick nine distinct points with integer coordinates in the three-dimensional space. Prove that there must exist two of these nine points—call them P and Q—such that the line segment PQ has a point (other than P and Q) on it with integer coordinates.

18. **(a)** Let p be a prime number, and x an integer not divisible by p. Prove that there exist non-zero integers a,b of absolute values less than $\sqrt{p}$ such that $p|(ax-b)$.

(b) Now assume that p is of the form $4k+1$. We know from number theory that in this case there exists an integer x such that $p|(x^2+1)$. Show that $p=a^2+b^2$ for some integers a,b.

19. Let $a,b\in\mathbb{N}$ with $\gcd(a,b)=1$. Use the pigeon-hole principle to prove that $ua+vb=1$ for some $u,v\in\mathbb{Z}$.

20. [*Chinese remainder theorem*] Let $m,n\in\mathbb{N}$ with $\gcd(m,n)=1$, $a\in\{0,1,2,\ldots,m-1\}$, and $b\in\{0,1,2,\ldots,n-1\}$. Prove that there exists an integer x such that x rem $m=a$, and x rem $n=b$.

21. Let ξ be an irrational number. Prove that given any real $\varepsilon>0$ (no matter how small), there exist integers a,b such that $0<a\xi-b<\varepsilon$.

22. Let $n\geqslant10$ be an integer. You choose n distinct elements from the set $\{1,2,3,\ldots,n^2\}$. Prove that there must exist two disjoint non-empty subsets of the chosen numbers, whose sums are equal.