CS21001 Discrete Structures
Tutorial 1

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1. Consider paths from $(0,0)$ to (n,n) in an $n \times n$ grid, that never cross the diagonal. Impose an additional constraint that these paths are not allowed to touch the main diagonal except only at the beginning and at the end. How many such constrained paths are there?

2. How many sorted arrays of size n are there if each element of the array is an integer in the range $1,2,3,\ldots,r$?

3. Prove the following logical deduction.

$$\begin{array}{l} (\neg p \vee q) \rightarrow r \\ r \rightarrow (s \vee t) \\ \neg(s \vee u) \\ t \rightarrow u \\ q \leftrightarrow v \\ v \veebar w \rightarrow \neg p \\ \hline \therefore \quad \neg w \end{array}$$

4. Encode the following logical statements using quantifiers, and conclude on the validity of the last statement.

   (a) Every athlete is strong.
   Everyone who is strong and intelligent will succeed in career.
   Hima is an athlete.
   Hima is intelligent.

   $\therefore$ Hima will succeed in career.

   (b) No man who is a candidate will be defeated if he is a good campaigner.
   Any man who runs for office is a candidate.
   Any candidate who is not defeated will be elected.
   Every man who is elected is a good campaigner.

   $\therefore$ Any man who runs for office will be elected if and only if he is a good campaigner.

5. Prove that the negation of the statement $\exists x\, \forall y\, \Big[P(x,y) \rightarrow \neg Q(y)\Big]$ is $\forall x\, \exists y\, \Big[P(x,y) \wedge Q(y)\Big]$.

6. Prove that $\forall x\, [P(x) \rightarrow (Q(x) \leftrightarrow R(x))]$ is equivalent to

$$\Big[\forall x\, \big[(P(x) \wedge Q(x)) \rightarrow R(x)\big]\Big] \wedge \Big[\forall x\, \big[(P(x) \wedge R(x)) \rightarrow Q(x)\big]\Big].$$

7. Prove that, for any prime p, $\sqrt{p}$ is irrational.

8. Prove that $\forall a,b,c \in \mathbb{N}\, \Big[a|(bc) \rightarrow [(a|b) \vee (\gcd(a,c) > 1)]\Big]$.

9. Prove that $\forall a,b,c \in \mathbb{N}\, \Big[(\gcd(a,b) = 1) \rightarrow \exists x \in \mathbb{N}\, [\gcd(a+bx,c) = 1]\Big]$.

10. Prove that the principle of mathematical induction implies the following.

    (a) $\mathbb{N}$ is well-ordered.

    (b) $\mathbb{Z}$ is well-ordered.

11. **(a)** Prove that $2^n < n! < 2^{n \log_2 n}$ for all $n \geqslant 4$.

    (b) Prove that for all $n \geqslant 4$, the n-th Catalan number satisfies $C_n \leqslant 2^{2n-4}$.

    (c) Prove that the harmonic numbers $H_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$ satisfy $\ln(n+1) \leqslant H_n \leqslant \ln n + 1$ for all $n \geqslant 1$.

12. Let $T(n)$ denote the number of disk movements performed by the following recursive algorithm for solving the three-peg Tower-of-Hanoi problem.

```
/* Move n disks from Peg A to Peg B using the auxiliary Peg C */
ToH ( n, A, B, C )
{
   if (n == 1) move the only disk from Peg A to Peg B.
   else {
      ToH(n-1,A,C,B);
      Move the largest disk from Peg A to Peg B.
      ToH(n-1,C,B,A);
   }
}
```

(a) Prove that $T(n) = 2^n - 1$.

(b) Prove that no algorithm can solve the problem in less than these many moves.

13. Prove the following assertions about Fibonacci numbers F_n, $n \geqslant 0$.

(a) $F_{m+n} = F_m F_{n+1} + F_{m-1} F_n$ for all $m \geqslant 1$ and $n \geqslant 0$.

(b) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$ for all $n \geqslant 1$.

(c) $\sum_{i=1}^{n} F_i = F_{n+2} - 1$ for all $n \geqslant 1$.

(d) $\sum_{i=0}^{n-1} F_{2i+1} = F_{2n}$ for all $n \geqslant 1$.

(e) $\sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1$ for all $n \geqslant 1$.

(f) For $n \geqslant 1$, inductively define $F_{-n} = F_{-n+2} - F_{-n+1}$. Prove that $F_{-n} = (-1)^{n+1} F_n$ for all $n \geqslant 1$.

(g) For all $m, n \geqslant 1$, if $m|n$, then $F_m | F_n$.

(h) For all $n \geqslant 1$, $\gcd(F_n, F_{n+1}) = \gcd(F_n, F_{n+2}) = \gcd(F_{n+1}, F_{n+2}) = 1$. (That is, any three consecutive Fibonacci numbers are coprime to one another.)

14. Prove that $\gcd(F_m, F_n) = F_{\gcd(m,n)}$ for all $m, n \geqslant 1$.

15. What does the following function return for integer inputs $m, n \geqslant 0$?

```
int f ( int m, int n )
{
   if ( (m == 0) || (n == 0) ) return 1;
   return f(m,n-1) + f(m-1,n) - 1;
}
```

16. The following function takes integer inputs $m, n \geqslant 0$. Determine the value of $g(3, n)$ as a function of n.

```
int g ( int m, int n )
{
   if ( (m == 0) || (n == 0) ) return 1;
   return g(m,n-1) + g(m-1,n);
}
```

17. What does the following function return on input n? Also argue that the function terminates for $n \geqslant 1$.

```
int h ( int n )
{
   if (n <= 0) return -1;      /* Error condition */
   if (n % 2 == 1) return 0;   /* n is odd */
   return 1 + h(n*(n+1)/2);    /* n is even */
}
```