CS21001 Discrete Structures, Autumn 2007

Tutorial 9: October 4 and 11, 2007

---

1 Use generating functions to solve the following recurrence relations.

(a) $a_0 = 1$, $a_n = 2a_{n-1}$ for $n \geqslant 1$.

Solution Let $a(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$. Then we have $a(x) = a_0 + (2a_0)x + (2a_1)x^2 + (2a_2)x^3 + \cdots + (2a_{n-1})x^n + \cdots = a_0 + 2xa(x)$, i.e., $(1-2x)a(x) = a_0 = 1$, i.e., $a(x) = \frac{1}{1-2x} = 1 + 2x + 2^2x^2 + 2^3x^3 + \cdots + 2^nx^n + \cdots$. Thus, $a_n = 2^n$ for all $n \in \mathbb{N}$.

(b) $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$.

Solution Let $F(x) = F_0 + F_1x + F_2x^2 + \cdots + F_nx^n + \cdots$. Then we have $F(x) = F_0 + F_1x + (F_1 + F_0)x^2 + (F_2 + F_1)x^3 + \cdots + (F_{n-1} + F_{n-2})x^n + \cdots = F_0 + F_1x + (F_1x^2 + F_2x^3 + \cdots + F_{n-1}x^n + \cdots) + (F_0x^2 + F_1x^3 + \cdots + F_{n-2}x^n + \cdots) = F_0 + F_1x + x(F(x) - F_0) + x^2F(x)$, i.e., $(1 - x - x^2)F(x) = F_0 + F_1x - F_0x = x$, i.e., $F(x) = \frac{x}{1-x-x^2} = \frac{x}{(1-\rho x)(1-\rho' x)} = \frac{1}{\rho-\rho'}\left(\frac{1}{1-\rho x} - \frac{1}{1-\rho' x}\right)$, where $\rho = \frac{1+\sqrt{5}}{2}$ is the golden ratio and $\rho' = \frac{1-\sqrt{5}}{2}$. We have $\rho - \rho' = \sqrt{5}$, and so $F(x) = \frac{1}{\sqrt{5}}\left[(1 + \rho x + \rho^2x^2 + \cdots + \rho^nx^n + \cdots) - (1 + \rho' x + \rho'^2x^2 + \cdots + \rho'^nx^n + \cdots)\right]$, i.e., $F_n = \frac{1}{\sqrt{5}}(\rho^n - \rho'^n)$ for all $n \in \mathbb{N}$.

(c) $a_0 = 1$, $a_1 = 3$, $a_n = 4a_{n-1} - 4a_{n-2}$ for all $n \geqslant 2$.

Solution Let $a(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$. Then $a(x) = a_0 + a_1x + (4a_1 - 4a_0)x^2 + (4a_2 - 4a_1)x^3 + \cdots + (4a_{n-1} - 4a_{n-2})x^n + \cdots = a_0 + a_1x + 4x(a(x) - a_0) - 4x^2a(x)$, i.e., $(1 - 4x + 4x^2)a(x) = a_0 + a_1x - 4a_0x = 1 - x$, i.e., $a(x) = \frac{1-x}{1-4x+4x^2} = \frac{1-x}{(1-2x)^2} = \frac{1}{2}\left[\frac{1+(1-2x)}{(1-2x)^2}\right] = \frac{1}{2}\left[\frac{1}{(1-2x)^2} + \frac{1}{1-2x}\right] = \frac{1}{2}[(1 + 2(2x) + 3(2x)^2 + 4(2x)^3 + \cdots + (n+1)(2x)^n + \cdots) + (1 + 2x + (2x)^2 + (2x)^3 + \cdots + (2x)^n + \cdots)]$, that is, $a_n = \frac{1}{2}(n + 1 + 1)2^n = (n + 2)2^{n-1}$ for all $n \in \mathbb{N}$.

Note: In all the above examples, the denominator turns out to be the *opposite* of the characteristic polynomial $\chi(x)$, i.e., the polynomial $x^k\chi(1/x)$, where k is the order of the recurrence. This is not a mere coincidence. Indeed the following general result establishes this connection.

2 Using generating functions, prove the master theorem for homogeneous linear recurrences of finite order with constant coefficients.

Solution Consider the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$ for $n \geqslant k$ with $a_0, a_1, \ldots, a_{k-1}$ supplied as initial conditions. As usual, let $a(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$. Using the recurrence relation in the coefficients of $x^k, x^{k+1}, x^{k+2}, \ldots$ yields

$$\begin{aligned}
a(x) &= a_0 + a_1x + \cdots + a_{k-1}x^{k-1} + \\
&\quad (c_1a_{k-1} + c_2a_{k-2} + \cdots + c_ka_0)x^k + \\
&\quad (c_1a_k + c_2a_{k-1} + \cdots + c_ka_1)x^{k+1} + \\
&\quad (c_1a_{k+1} + c_2a_k + \cdots + c_ka_2)x^{k+2} + \\
&\quad \cdots + \\
&\quad (c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k})x^n + \\
&\quad \cdots \\
&= a_0 + a_1x + \cdots + a_{k-1}x^{k-1} + \\
&\quad c_1(a_{k-1}x^k + a_kx^{k+1} + a_{k+1}x^{k+2} + \cdots + a_{n-1}x^n + \cdots) + \\
&\quad c_2(a_{k-2}x^k + a_{k-1}x^{k+1} + a_kx^{k+2} + \cdots + a_{n-2}x^n + \cdots) + \\
&\quad \cdots + \\
&\quad c_k(a_0x^k + a_1x^{k+1} + a_2x^{k+2} + \cdots + a_{n-k}x^n + \cdots)
\end{aligned}$$

$$
\begin{aligned}
&= a_0 + a_1x + \cdots + a_{k-1}x^{k-1} + \\
&\quad c_1x[a(x) - (a_0 + a_1x + \cdots + a_{k-2}x^{k-2})] + \\
&\quad c_2x^2[a(x) - (a_0 + a_1x + \cdots + a_{k-3}x^{k-3})] + \\
&\quad \cdots + \\
&\quad c_kx^ka(x).
\end{aligned}
$$

This implies that $(1 - c_1x - c_2x^2 - \cdots - c_kx^k)a(x) = p(x)$, i.e., $a(x) = \frac{p(x)}{q(x)}$, where $p(x)$ is a polynomial of degree $\leqslant k-1$ and $q(x) = x^k\chi(1/x)$ is the opposite of the characteristic polynomial. Note that the polynomial $p(x)$ is fully determined only by the initial conditions $a_0, a_1, \ldots, a_{k-1}$ (and the constant coefficients $c_1, c_2, \ldots, c_k$).

Let $r_1, r_2, \ldots, r_t$ be the roots of the characteristic equation with respective multiplicities $m_1, m_2, \ldots, m_t$. That is, $q(x) = (1 - r_1x)^{m_1}(1 - r_2x)^{m_2} \cdots (1 - r_tx)^{m_t}$. Since $\deg p(x) \leqslant k - 1$ and $\deg q(x) = k$, the partial fraction expansion of $a(x)$ is of the form

$$
a(x) = \sum_{i=1}^{t}\sum_{j=1}^{m_i} \frac{u_{i,j}}{(1 - r_ix)^j} = \sum_{i=1}^{t}\left(\frac{u_{i,1}}{(1 - r_ix)} + \frac{u_{i,2}}{(1 - r_ix)^2} + \cdots + \frac{u_{i,m_i}}{(1 - r_ix)^{m_i}}\right),
$$

where $u_{i,j}$ are (real or complex) constants. Using the generalized binomial theorem then yields

$$
\begin{aligned}
a(x) &= \sum_{i=1}^{t}\Bigg[u_{i,1}\Big(1 + (r_ix) + (r_ix)^2 + \cdots + (r_ix)^n + \cdots\Big) + \\
&\quad u_{i,2}\Big(1 + 2(r_ix) + 3(r_ix)^2 + \cdots + (n+1)(r_ix)^n + \cdots\Big) + \\
&\quad u_{i,3}\Big(1 + \frac{2\times 3}{2!}(r_ix) + \frac{3\times 4}{2!}(r_ix)^2 + \cdots + \frac{(n+1)(n+2)}{2!}(r_ix)^n + \cdots\Big) + \\
&\quad \cdots + \\
&\quad u_{i,m_i}\Big(1 + \frac{2\times 3\times\cdots\times m_i}{(m_i-1)!}(r_ix) + \frac{3\times 4\times\cdots\times(m_i+1)}{(m_i-1)!}(r_ix)^2 + \cdots + \\
&\qquad \frac{(n+1)(n+2)\cdots(n+m_i-1)}{(m_i-1)!}(r_ix)^n + \cdots\Big)\Bigg] \\
&= \sum_{i=1}^{t}\sum_{n\in\mathbb{N}}\left(u_{i,1} + (n+1)u_{i,2} + \frac{(n+1)(n+2)}{2!}u_{i,3} + \cdots + \frac{(n+1)(n+2)\cdots(n+m_i-1)}{(m_i-1)!}u_{i,m_i}\right)r_i^nx^n \\
&= \sum_{i=1}^{t}\sum_{n\in\mathbb{N}} u_i(n)r_i^nx^n \\
&= \sum_{n\in\mathbb{N}}(u_1(n)r_1^n + u_2(n)r_2^n + \cdots + u_t(n)r_t^n)x^n,
\end{aligned}
$$

where for each i, $u_i(n)$ is a polynomial in n of degree $\leqslant m_i - 1$. It follows that $a_n = u_1(n)r_1^n + u_2(n)r_2^n + \cdots + u_t(n)r_t^n$ for all $n \geqslant 0$. In particular, if all $m_i = 1$, i.e., if there are no repeated roots of the characteristic polynomial (so that $t = k$), each $u_i(n)$ is a constant, call it u_i, and we get $a_n = u_1r_1^n + u_2r_2^n + \cdots + u_kr_k^n$ for all $n \geqslant 0$.

3 Consider the recurrence relation

$$
a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k} + f_1(n) + f_2(n) \text{ for } n \geqslant k, \tag{1}
$$

with neither $f_1(n)$ nor $f_2(n)$ identically zero. Prescribe a method to solve this non-homogeneous recurrence relation. More precisely, identify a particular solution for this recurrence.

Solution Let $p_{1,n}$ be a particular solution of

$$
a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k} + f_1(n), \tag{2}
$$

and $p_{2,n}$ a particular solution of

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f_2(n). \quad (3)$$

But then $p_{1,n}+p_{2,n} = c_1(p_{1,n-1}+p_{2,n-1})+c_2(p_{1,n-2}+p_{2,n-2})+\cdots+c_k(p_{1,n-k}+p_{2,n-k})+f_1(n)+f_2(n)$, i.e., $p_n = p_{1,n}+p_{2,n}$ is a particular solution of the original recurrence relation (1). Therefore, we first obtain $p_{1,n}$ from (2) and then $p_{2,n}$ from (3). Let h_n denote a solution of the corresponding homogeneous equation. Then a general solution of (1) is $a_n = h_n + p_n = h_n + p_{1,n} + p_{2,n}$ for all $n \geqslant 0$.

4 Solve the recurrence relation $a_0 = 1$, $a_n = 2a_{n-1} + n + 2^n$ for all $n \geqslant 1$.

Solution Here $f_1(n) = n \times 1^n$ and $f_2(n) = 1 \times 2^n$. The characteristic equation is $x - 2 = 0$, which has only one simple root 2. We have particular solutions of the form $p_{1,n} = (u_1 n + u_0) \times n^0 \times 1^n = u_1 n + u_0$ and $p_{2,n} = v_0 \times n^1 \times 2^n$. Since $p_{1,n} = 2p_{1,n-1} + n$, we have $u_1 n + u_0 = 2(u_1(n-1) + u_0) + n$, i.e., $(u_1 + 1)n + (u_0 - 2u_1) = 0$, i.e., $u_1 + 1 = 0$ and $u_0 - 2u_1 = 0$, i.e., $u_1 = -1$ and $u_0 = -2$, i.e., $p_{1,n} = -(n + 2)$. On the other hand, $p_{2,n} = 2p_{2,n-1} + 2^n$, so that $v_0 n 2^n = 2v_0(n-1)2^{n-1} + 2^n$, i.e., $v_0 n = v_0(n-1) + 1$, i.e., $v_0 = 1$, i.e., $p_{2,n} = n2^n$. Thus the original recurrence relation has the particular solution $p_n = n2^n - (n + 2)$. The solution of the corresponding homogeneous equation is $h_n = w2^n$, i.e., the general solution is $a_n = h_n + p_n = w2^n + n2^n - (n + 2)$. Since $a_0 = 1 = w + 0 - 2$, we have $w = 3$, i.e., $a_n = (n + 3)2^n - (n + 2)$ for all $n \geqslant 0$.

5 Assume that $T(n)$ is an increasing function of n, that satisfies $T(n) = 25T(n/5) + 125n^d$ whenever $n = 5^t$ for $t \geqslant 1$. Determine the orders of $T(n)$ in the Θ-notation for $d = 1, 2, 3$.

Solution Here $e = \log_5 25 = 2$. Thus, for $d = 1$, $T(n) = \Theta(n^2)$, for $d = 2$, $T(n) = \Theta(n^2 \log n)$, and for $d = 3$, $T(n) = \Theta(n^3)$.

Additional exercises

6 Consider the proof of Exercise 2. If $u_{i,m_i} = 0$, then $q(x)$ is not divisible by $(1 - r_i x)^{m_i}$ (but by $(1 - r_i x)^{m_i'}$ for some $m_i' < m_i$). So we must have $u_{i,m_i} \neq 0$. But then $u_i(n)$ is of degree equal to $m_i - 1$.

(a) Find a recurrence relation for which some $u_i(n)$ has degree strictly smaller than $m_i - 1$. Your sequence may not correspond to the zero sequence (with all-zero initial values). Try to locate a sequence in which every term is non-zero.

(b) Resolve the fallacy.

7 Use generating functions to solve the following non-homogeneous recurrence relations.

(a) $a_0 = 1$, $a_n = 2a_{n-1} + 1$ for $n \geqslant 1$.
(b) $a_0 = 1$, $a_n = 2a_{n-1} + n$ for $n \geqslant 1$.
(c) $a_0 = 1$, $a_n = 2a_{n-1} + 2^n$ for $n \geqslant 1$.
(d) $a_0 = 1$, $a_1 = 2$, $a_n = a_{n-1} + 2a_{n-2} + n$ for $n \geqslant 2$.
(e) $a_0 = 1$, $a_1 = 2$, $a_n = a_{n-1} + 2a_{n-2} + 2^n$ for $n \geqslant 2$.

8 Using generating functions, prove the master theorem for linear non-homogeneous recurrence relations of constant degrees and with constant coefficients. Restrict the non-homogeneous part only to functions of the form $f(n) = s^n p(n)$ for a real number $s \neq 0$ and for a polynomial $p(n)$.

9 Define a recurrence relation and the requisite number of initial conditions for each of the following.

(a) The number of binary strings of length n, that do not contain three consecutive 0's.
(b) The number of binary strings of length n, that do not contain k consecutive 0's, where $k \in \mathbb{N}$ is a constant. (**Hint:** Look at the first occurrence of 1.)

10 Solve the following recurrence relations.

(a) $a_0 = 2$, $a_1 = 3$, $a_n = a_{n-1} + 12a_{n-2}$ for $n \geqslant 2$.
(b) $a_0 = 2$, $a_1 = 3$, $6a_n = a_{n-1} + 12a_{n-2}$ for $n \geqslant 2$.
(c) $a_0 = 2$, $a_1 = 3$, $a_2 = 4$, $a_n = a_{n-1} + 4a_{n-2} - 4a_{n-2}$ for $n \geqslant 3$.
(d) $a_0 = 2$, $a_1 = 3$, $a_2 = 4$, $a_3 = 5$, $a_n = 2a_{n-2} - a_{n-4}$ for $n \geqslant 4$.
(e) $a_0 = 2$, $a_1 = 3$, $a_2 = 4$, $a_3 = 5$, $a_n = 3a_{n-2} - 2a_{n-4}$ for $n \geqslant 4$.
(f) $a_0 = 2$, $a_n = 5a_{n-1} + (n^2 + n + 1)$ for $n \geqslant 1$.
(g) $a_0 = 2$, $a_n = 5a_{n-1} + (n^2 + n + 1)2^n$ for $n \geqslant 1$.
(h) $a_0 = 2$, $a_1 = 3$, $a_n = 2(a_{n-1} + a_{n-2} + 2^n)$ for $n \geqslant 2$.
(i) $a_0 = 2$, $a_1 = 3$, $a_n = 4(a_{n-1} - a_{n-2} + 2^n)$ for $n \geqslant 2$.
(j) $a_0 = 2$, $a_1 = 3$, $a_n = 4(a_{n-1} - a_{n-2} + n2^{2n-1})$ for $n \geqslant 2$.

11 Solve the following recurrence relations.

(a) $a_0 = 2$, $a_n = 2a_{n-1} + 2^n + n^2$ for all $n \geqslant 1$.
(b) $a_0 = 2$, $a_1 = 3$, $a_n = 2a_{n-1} - a_{n-2} + 2^n + n^2$ for all $n \geqslant 2$.
(c) $a_0 = 2$, $a_1 = 3$, $a_n = a_{n-2} + 2^n + n3^n + n^2 4^n$ for all $n \geqslant 2$.

12 Reduce the following recurrence relations to standard forms and solve.

(a) $a_0 = 2$, $a_1 = 3$, $a_n = a_{n+2} - a_{n+1} - n$ for all $n \geqslant 0$.
(b) $a_0 = 2$, $a_1 = 3$, $4a_{n+1} + 8a_n - 5a_{n-1} = 2^n$ for all $n \geqslant 1$.
(c) $a_0 = 2$, $a_1 = 3$, $a_n = a_{n-1} + 12(a_{n-2} + 2^{n-2} + 1)$ for all $n \geqslant 2$.
(d) $a_0 = 2$, $a_n^3 = a_{n-1}(3a_n^2 - 3a_n a_{n-1} + a_{n-1}^2) + n^3$ for all $n \geqslant 1$.
(e) $a_0 = 2$, $a_1 = 3$, $2a_n a_{n-2} - 2a_{n-1}^2 - 3a_{n-1}a_{n-2} = 0$ for all $n \geqslant 2$.
(f) $a_0 = 2$, $a_1 = 3$, $2^{a_n} = 4^n \times 16^{a_{n-2}}$ for all $n \geqslant 2$.

13 Find big-Θ estimates for the following positive-integer-valued increasing functions $f(n)$.

(a) $f(n) = 125f(n/4) + 2n^3$ whenever $n = 4^t$ for $t \geqslant 1$.
(b) $f(n) = 125f(n/5) + 2n^3$ whenever $n = 5^t$ for $t \geqslant 1$.
(c) $f(n) = 125f(n/6) + 2n^3$ whenever $n = 6^t$ for $t \geqslant 1$.

14 Let $f(n)$ be an increasing positive-real-valued function of a non-negative integer variable n. Give a big-Θ estimate of $f(n)$ for each of the following cases.

(a) $f(n) = 2f(\sqrt{n}) + 1$ whenever n is a perfect square larger than 1.
(b) $f(n) = 2f(\sqrt{n}) + \log n$ whenever n is a perfect square larger than 1.
(c) $f(n) = 2f(\sqrt{n}) + \log^2 n$ whenever n is a perfect square larger than 1.
(d) $f(n) = af(\sqrt[b]{n}) + c(\log n)^d$ whenever n is a perfect b-th power larger than 1. Here $a, b \in \mathbb{N}$, $a \geqslant 1$, $b \geqslant 2$, $c, d \in \mathbb{R}$, $c > 0$ and $d \geqslant 0$.