CS21001 Discrete Structures, Autumn 2007

Tutorial 8: September 27, 2007

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1 Let $S(n,k)$ denote Stirling numbers of the second kind and $\left\{ {n \atop k} \right\}$ denote Stirling subset numbers. Prove that $S(n,k) = \left\{ {n \atop k} \right\}$ for all $n, k \in \mathbb{N}$.

Solution We proceed by induction on n. For $n = 0$, we have $x^n = x^0 = 1 = x^{\underline{0}}$, i.e., $S(n,0) = 1$ and $S(n,k) = 0$ for all $k \geqslant 1$. On the other hand, by definition $\left\{ {0 \atop 0} \right\} = 1$ and $\left\{ {0 \atop k} \right\} = 0$ for all $k \geqslant 1$.

Now suppose $n \geqslant 1$ and $S(n-1,k) = \left\{ {n-1 \atop k} \right\}$ for all $k \in \mathbb{N}$. But then $x^{n-1} = \sum_{k\in\mathbb{N}} S(n-1,k)x^{\underline{k}} = \sum_{k\in\mathbb{N}} \left\{ {n-1 \atop k} \right\} x^{\underline{k}}$. Multiplication by x yields $x^n = \sum_{k\in\mathbb{N}} \left\{ {n-1 \atop k} \right\} x^{\underline{k}} \times x$. Now $x^{\underline{k}} \times x = x^{\underline{k}} \times (x-k) + x^{\underline{k}} \times k = x^{\underline{k+1}} + kx^{\underline{k}}$. Therefore, $x^n = \sum_{k\in\mathbb{N}} \left\{ {n-1 \atop k} \right\} (x^{\underline{k+1}} + kx^{\underline{k}}) = 0 \times x^{\underline{0}} + \sum_{n\geqslant 1} \left(\left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\} \right) x^{\underline{k}} = \sum_{n\in\mathbb{N}} \left\{ {n \atop k} \right\} x^{\underline{k}}$, i.e., $S(n,k) = \left\{ {n \atop k} \right\}$ for all $k \in \mathbb{N}$.

2 **(a)** Let C_n, $n \in \mathbb{N}$, denote the sequence of Catalan numbers, and let $C(x) = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n + \cdots$ the corresponding power series. Prove that $C(x) = \frac{1-\sqrt{1-4x}}{2x}$.

Solution We have $C(x)C(x) = C_0C_0 + (C_0C_1 + C_1C_0)x + (C_0C_2 + C_1C_1 + C_2C_0)x^2 + \cdots + (C_0C_n + C_1C_{n-1} + \cdots + C_nC_0)x^n + \cdots = C_1 + C_2x + C_3x^2 + \cdots + C_{n+1}x^n + \cdots$, i.e., $xC(x)^2 = -C_0 + C(x) = -1 + C(x)$, i.e., $xC(x)^2 - C(x) + 1 = 0$. Therefore, $C(x) = \frac{1\pm\sqrt{1-4x}}{2x}$. Taking the plus sign of the square-root gives a Laurent series with a non-zero coefficient of x^{-1}, whereas $C(x)$ is a power series. Therefore, $C(x) = \frac{1-\sqrt{1-4x}}{2x}$.

(b) Deduce that $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Solution Expand $\sqrt{1-4x}$ by the generalized binomial theorem: $C_n = \frac{-(-1)^{n+1}\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-n\right)4^{n+1}}{(n+1)!\times 2} = \frac{2^n\times 1\times 3\times 5\times\cdots\times(2n-1)}{(n+1)!} = \frac{1}{n+1} \times \frac{(2n)!}{n!n!} = \frac{1}{n+1}\binom{2n}{n}$.

3 **(a)** Let A_n denote the set of all strings with n left parentheses and n right parentheses such that the parentheses are properly balanced (i.e., nested). For example, take $n = 3$. The strings $()()()$, $(())()$, $()(())$, $(()())$, and $((()))$ are all the strings of length $2n = 6$ with balanced parentheses. On the other hand, $())(()$ is not a member of A_3, since the second right parenthesis is not balanced by a left parenthesis, and the second left parenthesis is not balanced by a right parenthesis. Prove that the size of A_n is C_n.

Solution Consider an explicit parenthesizing of a product of $n+1$ matrices $M_0, M_1, \ldots, M_n$, i.e., a pair of matching parentheses identifies each of the n products taken (including the outermost product). Suppose also that each multiplication is shown explicitly by a sign, say $\times$. First delete all the matrix arguments and all the left parentheses, then replace each multiplication sign by a left parenthesis. This results in a balanced expression with n left and n right parentheses. For example, take $n = 3$. The product $((M_0 \times M_1) \times (M_2 \times M_3))$ translates to the string $()(())$, whereas the product $(M_0 \times (M_1 \times (M_2 \times M_3)))$ translates to $((()))$. Convince yourself that this association provides a one-to-one correspondence between the set of all fully parenthesized products of $M_0, M_1, \ldots, M_n$ and the set A_n.

(b) Let there be $2n$ distinct points on a circle. In how many ways can you connect these points to n chords so that no two chords intersect inside the circle?

Solution This count is again C_n. Name the points $P_1, P_2, \ldots, P_{2n}$ clockwise along the circle (starting from an arbitrary point). Let $\alpha = a_1a_2\ldots a_{2n}$ be a string of length $2n$ with balanced parenthesis. For each matching pair of left parenthesis a_i and right parenthesis a_j, join the points P_i and P_j to form a chord. Since α is balanced, no two chords intersect inside the circle. Convince yourself that this association provides a bijective correspondence between the set of all balanced strings of n left and n right parentheses and the set of all non-intersecting chord constructions.

Additional exercises

4 Let n be a positive integer. By $\phi(n)$, denote the number of integers a between 1 and n (both inclusive) with $\gcd(a,n) = 1$. Assume that $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ be the prime factorization of n (with each $e_i > 0$). Use the principle of inclusion and exclusion to derive the formula $\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_r}\right)$.

5 Let A_n denote the set of n-digit integers, i.e., integers in the range 10^{n-1} to $10^n - 1$. How many elements of A_n do not contain repeated digits?

6 Let $s(n,k)$ denote Stirling numbers of the first kind and $\left[{n \atop k}\right]$ denote Stirling cycle numbers. Show that $\left[{n \atop k}\right] = (-1)^{n-k} s(n,k)$ for all $n, k \in \mathbb{N}$.

7 Without using generating functions, prove that $C_n = \frac{1}{n+1}\binom{2n}{n}$.

8 Prove the following identities.

(a) $\left\{{n \atop 1}\right\} = 1$, $\left\{{n \atop 2}\right\} = 2^{n-1} - 1$, $\left\{{n \atop n-1}\right\} = \binom{n}{2}$, and $\left\{{n \atop n}\right\} = 1$.

(b) $\left[{n \atop 1}\right] = (n-1)!$, $\left[{n \atop 2}\right] = (n-1)!\,H_{n-1}$, $\left[{n \atop n-3}\right] = \binom{n}{2}\binom{n}{4}$, $\left[{n \atop n-2}\right] = \frac{1}{4}(3n-1)\binom{n}{3}$, $\left[{n \atop n-1}\right] = \binom{n}{2}$, and $\left[{n \atop n}\right] = 1$.

(c) $\sum_{k=0}^{n} \left[{n \atop k}\right] = n!$.

9 The *Bell number* B_n is defined as $B_n = \sum_{k=0}^{n} \left\{{n \atop k}\right\}$ and equals the total number of partitions of a set of n elements. Prove that $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$.

10 Prove that each of the following equals C_n.

(a) The number of sequences $a_1 a_2 \ldots a_{2n}$ of length $2n$ with each $a_i \in \{1, -1\}$, $\sum_{i=1}^{2n} a_i = 0$, and $\sum_{i=1}^{j} a_i \geqslant 0$ for all $j \in \{1, 2, \ldots, 2n\}$.

(b) The number of (rooted) binary trees with n internal vertices. (An internal node has at least one child.)

(c) The number of paths from the lower left corner to the upper right corner in an $n \times n$ grid, that do not rise above the grid diagonal connecting the two corners mentioned above.

(d) The number of ways an $(n+2)$-gon can be cut into triangles.

11 Use the theory of generating functions to find explicit formulas for a_n defined recursively as

(a)
$$\begin{aligned} a_0 &= 1, \\ a_n &= a_{n-1} + a_{n-2} + a_{n-3} + \cdots + a_0 \quad \text{for all } n \geqslant 1. \end{aligned}$$

(b)
$$\begin{aligned} a_0 &= 1, \\ a_n &= a_{n-1} + 2a_{n-2} + 3a_{n-3} + \cdots + n a_0 \quad \text{for all } n \geqslant 1. \end{aligned}$$

12 Let $a(x), b(x), c(x)$ be power series. Prove that $a(x)(b(x) + c(x)) = a(x)b(x) + a(x)c(x)$.

13 The *formal derivative* $a'(x)$ of a power series $a(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$ is defined as the power series $a'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots$. For power series expressions $a(x), b(x)$, prove that

(a) $(a(x) + b(x))' = a'(x) + b'(x)$.

(b) $(a(x)b(x))' = a'(x)b(x) + a(x)b'(x)$.

14 Determine how to add, subtract, and multiply two Laurent series and invert a non-zero Laurent series.

A primer on power series and Laurent series

A *power series* $a(x)$ in one variable (or indeterminate) x is an infinite expression of the form $a(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$, where the coefficients $a_0, a_1, a_2, \ldots$ are taken from a standard set like $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, or $\mathbb{Z}$. In order to study sequences and generating functions, we need to work with these infinite expressions. In this (and many other) contexts, a power series is taken as *formal*. This means that we do not bother about the convergence issues of these series. Indeed, we do not need to assign numerical values to the variable x. The set of all formal power series over a set (a field or more generally a ring) A is denoted by $A[[x]]$.

We first define the standard arithmetic operations on two power series $a(x) = a_0 + a_1x + a_2x^2 + \cdots$ and $b(x) = b_0 + b_1x + b_2x^2 + \cdots$.

$$\begin{aligned} a(x) + b(x) &= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_n + b_n)x^n + \cdots, \\ a(x)b(x) &= (a_0b_0) + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \cdots + \\ & \quad\ (a_0b_n + a_1b_{n-1} + \cdots + a_nb_0)x^n + \cdots. \end{aligned}$$

The additive inverse of $a(x)$ is the power series

$$-a(x) = (-a_0) + (-a_1)x + (-a_2)x^2 + \cdots + (-a_n)x^n + \cdots.$$

The multiplicative inverse of $a(x)$ is defined as the (unique) power series $u(x)$ satisfying $a(x)u(x) = 1$, provided that such a series exists. The following result provides a necessary and sufficient condition for the invertibility of a power series.

Proposition A power series $a(x) = a_0 + a_1x + a_2x^2 + \cdots \in A[x]$ is invertible if and only if a_0 is invertible in A. In particular, if A is a field, then $a(x)$ is invertible if and only if $a_0 \neq 0$.

Proof [only if] Let $u(x) = u_0 + u_1x + u_2x^2 + \cdots$ be the inverse of $a(x)$. Then $a_0u_0 = 1$, i.e., a_0 is invertible in A (having the inverse $a_0^{-1} = u_0$).

[if] Suppose that a_0 is invertible in A. We inductively compute elements $u_0, u_1, u_2, \ldots \in A$ such that $a(x)u(x) = 1$. First take $u_0 = a_0^{-1}$. Then assume that $u_0, u_1, \ldots, u_{n-1}$ are already computed for some $n \geqslant 1$. The condition $a(x)u(x) = 1$ demands $a_0u_n + a_1u_{n-1} + \cdots + a_{n-1}u_1 + a_nu_0 = 0$. Since a_0 is invertible in A and since $u_0, u_1, \ldots, u_{n-1}$ are available, we obtain u_n as $u_n = -a_0^{-1}(a_1u_{n-1} + \cdots + a_{n-1}u_1 + a_nu_0)$. This completes the inductive construction. •

It is interesting to highlight that the only invertible polynomials are non-zero constants. On the other hand, a non-constant power series may have an inverse. For example, $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \cdots + (-1)^nx^n + \cdots$.

The only invertible integers are ± 1. In order to make every non-zero integer invertible, we embedded $\mathbb{Z}$ in a bigger structure $\mathbb{Q}$. Analogously, in order to invert every non-zero polynomial, we invent the notion of rational functions, i.e., expressions of the form $f(x)/g(x)$ for polynomials $f(x), g(x)$ with $g(x) \neq 0$. For power series too, we define the structure

$$A((x)) = \{a(x)/b(x) \mid a(x), b(x) \in A[[x]],\ b(x) \neq 0\}.$$

Elements of $A((x))$ are, therefore, quotients of power series expressions. It turns out that these quotients have an easier description. For simplicity, we restrict our study only to the case that A is a field (like $\mathbb{R}, \mathbb{C}, \mathbb{Q}$) in which every non-zero element is invertible.

Let $a(x), b(x) \in A[[x]]$ with $b(x) \neq 0$. If $b(x)$ itself is invertible in $A[[x]]$, then $b(x)^{-1} = u(x) \in A[[x]]$ and $a(x)/b(x) = a(x)b(x)^{-1} = a(x)u(x)$ is a power series too. So assume that $b(x)$ is non-invertible in $A[[x]]$, i.e., $b(x) = b_rx^r + b_{r+1}x^{r+1} + b_{r+2}x^{r+2} + \cdots$ with $b_r \neq 0$ for some $r \geqslant 1$. We have $b(x) = x^rc(x)$, where $c(x) = b_r + b_{r+1}x + b_{r+2}x^2 + \cdots$. Since $b_r \neq 0$, $c(x)$ is invertible in $A[[x]]$. Write $c(x)^{-1} = u(x) = u_0 + u_1x + u_2x^2 + \cdots$. Then $b(x)^{-1} = x^{-r}u(x)$ and $a(x)/b(x) = x^{-r}a(x)u(x)$. Let $a(x)u(x) = v(x) = v_0 + v_1x + v_2x^2 + \cdots$. Then $a(x)/b(x) = x^{-r}v(x) = v_0x^{-r} + v_1x^{-r+1} +$

$v_2x^{-r+2} + \cdots + v_{r-1}x^{-1} + v_r + v_{r+1}x + v_{r+2}x^2 + \cdots + v_{r+n}x^n + \cdots$. That is, $a(x)/b(x)$ is a power series plus a *finite* number of terms involving negative powers of x. Such a series is called a *Laurent series*, and $A((x))$ turns out to be the set of all Laurent series expressions, i.e.,

$$\begin{aligned} A((x)) &= \Big\{a_{-r}x^{-r} + a_{-r+1}x^{-r+1} + \cdots + a_{-1}x^{-1} + a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \mid \\ &\qquad r \in \mathbb{N}, a_i \in A \text{ for all } i = -r, -r+1, \ldots, -1, 0, 1, 2, \ldots\Big\}. \end{aligned}$$

One can readily extend arithmetic operations (addition, multiplication, negation, and inverse) to Laurent series expressions. Indeed, if A is a field, then so also is $A((x))$.