# CS21001 Discrete Structures, Autumn 2007

## Tutorial 7: September 06, 2007

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**1** Let $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geqslant 0\}$. In this exercise, we show that the sets $[0,1)$, $\mathbb{R}_+$, and $\mathbb{R}$ are equinumerous to one another. To that effect, we propose explicit bijections among them.

**(a)** Provide an explicit bijection $[0,1) \to \mathbb{R}_+$.

*Solution* Take $f : [0,1) \to \mathbb{R}_+$ as $f(x) = \dfrac{x}{1-x}$. Some other simple examples are $f(x) = -\log(1-x)$, $f(x) = \tan \frac{\pi x}{2}$, $f(x) = \tan \frac{\pi x}{x+1}$.

**(b)** Provide an explicit bijection $\mathbb{R}_+ \to \mathbb{R}$.

*Solution* For $n \in \mathbb{Z}$, let $I_n$ denote the real interval $[n, n+1)$. We have the disjoint unions $\mathbb{R}_+ = \bigcup_{n\in\mathbb{N}} I_n$ and $\mathbb{R} = \bigcup_{n\in\mathbb{Z}} I_n$. Define a bijection $g : \mathbb{R}_+ \to \mathbb{R}$ as follows. Take $x \in \mathbb{R}_+$. Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of $x$. We have $x \in I_n$, where $n = \lfloor x \rfloor$. If $n = 2k$ (even), define $g(x) = k + \{x\}$. If $n = 2k+1$ (odd), define $g(x) = -(k+1) + \{x\}$. More intuitively, $[0,1)$ is relocated to $[0,1)$, $[1,2)$ to $[-1,0)$, $[2,3)$ to $[1,2)$, $[3,4)$ to $[-2,-1)$, $[4,5)$ to $[2,3)$, $[5,6)$ to $[-3,-2)$, and so on.

**(c)** Conclude that the interval $[0,1)$ is equinumerous with the entire real line $\mathbb{R}$.

*Solution* The function $g \circ f$ is an explicit bijection $[0,1) \to \mathbb{R}$.

**2** Let $A$ be a set equinumerous with $\mathbb{R}$ and $B$ a set equinumerous with $\mathbb{N}$. Prove that the Cartesian product $A \times B$ is equinumerous with $\mathbb{R}$.

*Solution* There exist a bijection $f : A \to [0,1)$ and a bijection $g : B \to \mathbb{Z}$. But then $h = f \times g : A \times B \to [0,1) \times \mathbb{Z}$, $(a,b) \mapsto (f(a), g(b))$ is a bijection. Finally, the function $h' : [0,1) \times \mathbb{Z} \to \mathbb{R}$ taking $(a,n) \mapsto a + n$ is also a bijection. Thus, $h' \circ h$ is a bijection between $A \times B$ and $\mathbb{R}$.

(Remark: We have proved that $c \times \aleph_0 = c$.)

**3** Prove that the real interval $[0,1)$ is equinumerous with the unit square $S = \{x + \mathrm{i}y \mid x, y \in [0,1)\}$ in the complex plane.

*Solution* The canonical inclusion map $f : [0,1) \to S$, $x \mapsto x + \mathrm{i}0$, is injective. For constructing an injective map $g : S \to [0,1)$, let us plan to represent each real number with a terminating decimal expansion as having an infinite number of trailing 0's (instead of an infinite number of trailing 9's). Take $x + \mathrm{i}y \in S$, and let $x = 0.x_1x_2x_3\ldots$ and $y = 0.y_1y_2y_3\ldots$ be the decimal expansions of $x$ and $y$. Define $g(x + \mathrm{i}y) = 0.x_1y_1x_2y_2x_3y_3\ldots$. It is easy to check that this $g$ is an injective map. (Note that $g$ is *not* surjective. For example, the real number $0.4509090909\ldots$ does not have a pre-image.)

(Remark: We have proved that $c \times c = c$.)

**4** Provide a diagonalization argument to prove that the set $A$ of all infinite bit sequences is uncountable.

*Solution* Suppose that $A$ is countable. Then there exists a bijective map $f : \mathbb{N} \to A$. Denote the sequence $f(n)$ as $a_{n0}a_{n1}a_{n2}\ldots$, where each $a_{ij}$ is a bit. Construct an infinite bit sequence $b = b_0b_1b_2\ldots$ as $b_n = \begin{cases} 1 & \text{if } a_{nn} = 0, \\ 0 & \text{if } a_{nn} = 1. \end{cases}$ Since $f$ is surjective, $b = f(n)$ for some $n \in \mathbb{N}$. But $b$ and $f(n)$ differ in the $n$-th bit position and so cannot be equal, a contradiction.

**5** Prove that the set $B$ of all infinite subsets of $\mathbb{N}$ is uncountable.

*Solution* The power set of $\mathbb{N}$ is uncountable. In Tutorial 6, we have proved that the set $A$ of all finite subsets of $\mathbb{N}$ is countable. If $B$ is countable too, then the union $\mathcal{P}(\mathbb{N}) = A \cup B$ also becomes countable. So $B$ has to be uncountable.

**Additional exercises**

**6** Provide explicit bijections between the following pairs of sets.

**(a)** The sets $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$.
**(b)** The set of rational numbers in the real interval $[0, 1)$ and the set $\mathbb{Q}$ of all rational numbers.
**(c)** The set of irrational numbers in the real interval $[0, 1)$ and the set of all irrational numbers.
**(d)** The real interval $(0, 1)$ and $\mathbb{R}$.
**(e)** The real intervals $[0, 1)$ and $[a, b)$ for any $a, b \in \mathbb{R}$, $a < b$.
**(f)** The real intervals $[0, 1)$ and $(0, 1)$.
**(g)** The real intervals $[0, 1]$ and $(0, 1)$.

**7 (a)** Prove that the set $\mathbb{Q}[[X]]$ of all power series with rational coefficients is uncountable.
**(b)** Prove that the set $\mathbb{Q}(X) = \{f(X)/g(X) \mid g(X) \neq 0\}$ of all rational functions with rational coefficients is countable.
**(c)** Conclude that $\mathbb{Q}[[X]]$ contains a power series which is not the power series expansion of any rational function in $\mathbb{Q}(X)$. Can you identify any such power series explicitly?

**8 (a)** Let $A$ be a finite set of size $\geqslant 2$. Prove that the set of all functions $\mathbb{N} \to A$ is uncountable.
**(b)** Prove that the set of all functions $\mathbb{N} \to \{0, 1\}$ is equinumerous with $\mathbb{R}$ (i.e., $2^{\aleph_0} = c$).

**9** Let $A, B$ be sets, where $A$ is equinumerous with $\mathbb{R}$ and $B$ is equinumerous with $\mathbb{N}$. Prove that $A \cup B$ is equinumerous with $\mathbb{R}$. (This means $c + \aleph_0 = c$.)

**10** Prove that the union of two sets each equinumerous with $\mathbb{R}$ is again equinumerous with $\mathbb{R}$ (i.e., $c + c = c$).

**11** Prove that the set of all permutations of $\mathbb{N}$ is not countable. (One can show that $\aleph_0! = c$.)

**12** Let $A$ be a set of size $\geqslant 2$ ($A$ may be infinite). Modify the diagonalization proof to establish that there cannot exist a bijection between $A$ and the set of all *non-empty* subsets of $A$.

**13** Let $A = \{1, 2, 3, \ldots, 2n\}$ for some $n \geqslant 1$, and let $S$ be an arbitrary subset of $A$ of size $n + 1$. Prove that

**(a)** $A$ contains elements $x_1, y_1$ with $y_1 - x_1 = 1$.
**(b)** $A$ contains elements $x_2, y_2$ with $y_2 - x_2 = n$.
**(c)** $A$ contains elements $x_3, y_3$ with $\gcd(x_3, y_3) = 1$.

**14 (a)** Let $A$ be a finite set and $f : \mathbb{N} \to A$ a function. Take any $a \in A$ and denote $a_n = f^n(a)$ for all $n \in \mathbb{N}$. Prove that the sequence $a_0, a_1, a_2, \ldots$ of elements of $A$ must be (eventually) periodic.

**(b)** Let $n \in \mathbb{N}$, $n \geqslant 2$, and $a \in \mathbb{Z}$ with $\gcd(a, n) = 1$. Prove that $a^h \equiv 1 \pmod{n}$ for some $h \geqslant 1$.

**(c)** Prove that for every $n \in \mathbb{N}$, $n \neq 0$, there exists an $m \in \mathbb{N}$, $m \neq 0$, such that $n$ divides the Fibonacci number $F_m$.

**15** Suppose that you eat 132 idlis in a total of 77 consecutive days such that

1. You eat only a whole number of idlis per day.
2. You eat at least one idli in each day.

Show that there exists a set of consecutive days during which you eat a total of exactly 21 idlis.

**16** Suppose that acquaintance is a symmetric relation, i.e., two persons either know each other or are strangers to each each other. Prove that in any group of $n$ persons, $n \geqslant 2$, there exist two persons having the same number of acquaintances (in that group).

**17** Let $n$ be an odd positive integer and $a_1, a_2, \ldots, a_n$ a permutation of $1, 2, \ldots, n$. Prove that the product $(a_1 - 1)(a_2 - 2) \cdots (a_n - n)$ must be even.