CS21001 Discrete Structures, Autumn 2007

Tutorial 5: August 23, 2007

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1 In this exercise, we plan to construct a well-ordering of $A = \mathbb{N} \times \mathbb{N}$.

(a) First define a relation ρ on A as $(a, b)\ \rho\ (c, d)$ if and only if $a \leqslant c$ or $b \leqslant d$. Prove or disprove: ρ is a well-ordering of A.

Solution No. Indeed, ρ is not at all a partial order, since it is not antisymmetric: we have both $(1, 2)\ \rho\ (2, 1)$ and $(2, 1)\ \rho\ (1, 2)$, but $(1, 2) \neq (2, 1)$.

(b) Next define a relation σ on A as $(a, b)\ \sigma\ (c, d)$ if and only if $a \leqslant c$ and $b \leqslant d$. Prove or disprove: σ is a well-ordering of A.

Solution No. One can easily check that σ is a partial order on A. However, it is not a total order (and hence cannot be a well-ordering of A): the pairs $(1, 2)$ and $(2, 1)$ are, for example, not comparable.

(c) Finally, define a relation $\leqslant_L$ on A as $(a, b) \leqslant_L (c, d)$ if either (i) $a < c$ or (ii) $a = c$ and $b \leqslant d$. Prove that $\leqslant_L$ is a partial order on A.

Solution By Condition (ii), $(a, b) \leqslant_L (a, b)$. Now suppose that $(a, b) \leqslant_L (c, d)$ and $(c, d) \leqslant_L (a, b)$. If $a < c$, we cannot have $(c, d) \leqslant_L (a, b)$. Similarly, if $c < a$, we cannot have $(a, b) \leqslant_L (c, d)$. So $a = c$. But then $b \leqslant d$ and $d \leqslant b$, i.e., $b = d$. Finally, suppose that $(a, b) \leqslant_L (c, d)$ and $(c, d) \leqslant_L (e, f)$. Then $a \leqslant c$ and $c \leqslant e$. If $a < c$ or $c < e$, then $a < e$. On the other hand, if $a = c = e$, we must have $b \leqslant d$ and $d \leqslant f$. But then $b \leqslant f$.

(d) Prove that $\leqslant_L$ is a total order on A.

Solution Take any (a, b) and (c, d) in A. If $a < c$, then $(a, b) \leqslant_L (c, d)$. If $a > c$, then $(c, d) \leqslant_L (a, b)$. Finally, suppose that $a = c$. Since either $b \leqslant d$ or $d \leqslant b$, we have either $(a, b) \leqslant_L (c, d)$ or $(c, d) \leqslant_L (a, b)$.

(e) Is A well-ordered under $\leqslant_L$?

Solution Yes. Let S be a non-empty subset of A. Take $X = \{a \in \mathbb{N} \mid (a, b) \in A \text{ for some } b \in \mathbb{N}\}$. Since S is non-empty, X is non-empty too and contains a minimum element; call it x. For this x, let $Y = \{b \in \mathbb{N} \mid (x, b) \in S\}$. Since Y is a non-empty subset of $\mathbb{N}$, it contains a minimum element; call it y. It is now an easy check that (x, y) is a minimum element of S.

(f) Prove or disprove: An infinite subset of A may contain a maximum element.

Solution True. The infinite subset $\{(1, b) \mid b \in \mathbb{N}\} \cup \{(2, 1)\}$ of A contains the maximum element $(2, 1)$.

Note: The ordering $\leqslant_L$ on $\mathbb{N} \times \mathbb{N}$ described in this exercise is called the *lexicographic ordering*, since this is how you sort two-letter words in a dictionary. One can readily generalize this ordering to $\mathbb{N}^n$ for any $n \geqslant 3$.

2 Give an example of a poset A and a non-empty subset S of A such that S has lower bounds in A, but glb(S) does not exist.

Solution Take $A = \mathbb{Q}$ under the standard $\leqslant$ on rational numbers. Also take $S = \{x \in \mathbb{Q} \mid x^2 > 2\}$. Every rational number $< \sqrt{2}$ is a lower bound on S. Since $\sqrt{2}$ is irrational, glb(S) does not exist.

Another example: Take A to be the set of all irrational numbers between 1 and 5, and S to be the set of all irrational numbers between 2 and 3.

A simpler (but synthetic) example: Take $A = \{a, b, c, d\}$ and the relation

$$\rho = \{(a, a), (a, c), (a, d), (b, b), (b, c), (b, d)(c, c), (d, d)\}$$

on A. The subset $S = \{c, d\}$ of A has two lower bounds a and b, but these bounds are not comparable to one another.

Additional exercises

3 A *string* is a finite ordered sequence of symbols from a finite alphabet. We start with a predetermined total ordering of the alphabet and then define the usual dictionary order on strings. Prove that this dictionary order (again called *lexicographic ordering*) is a total ordering. Is it also a well-ordering?

4 Define a relation $\leqslant_{DL}$ on $A = \mathbb{N} \times \mathbb{N}$ as follows. Take $(a,b),(c,d) \in A$ and call $(a,b) \leqslant_{DL} (c,d)$ if either (i) $a+b < c+d$, or (ii) $a+b = c+d$ and $a \leqslant c$.

(a) Prove that $\leqslant_{DL}$ is a partial order on A.

(b) Prove that $\leqslant_{DL}$ is a total order on A.

(c) Is A well-ordered under $\leqslant_{DL}$?

(d) Prove or disprove: An infinite subset of A may contain a maximum element.

Note: The ordering $\leqslant_{DL}$ on A is called the *degree-lexicographic ordering*. Identify $(a,b) \in A$ with the monomial X^aY^b. First, order monomials with respect to their degrees. For two monomials of the same degree, apply lexicographic ordering. For example, $XY^3 \leqslant_{DL} Y^5$ and $XY^3 \leqslant_{DL} X^2Y^2$.

5 Generalize the degree-lexicographic ordering on $\mathbb{N}^n$ for any fixed $n \geqslant 3$.

6 Consider the following relation ρ on the set A of all positive rational numbers. Take $a/b, c/d \in A$ with $\gcd(a,b) = \gcd(c,d) = 1$. Call $(a/b)\,\rho\,(c/d)$ if and only if either (i) $a+b < c+d$ or (ii) $a+b = c+d$ and $a \leqslant c$. Prove that ρ is a total order. Prove that A is well-ordered by ρ.

7 Define a well-ordering of $\mathbb{Q}$.

8 Let A be the set of all functions $\mathbb{N} \to \mathbb{N}$. For $f, g \in A$, define $f \leqslant g$ if and only if $f(n) \leqslant g(n)$ for all $n \in \mathbb{N}$. prove that $\leqslant$ is a partial order on A. Is $\leqslant$ also a total order?

9 Let A be the set of all functions $\mathbb{N} \to \mathbb{N}$.

(a) Define a relation Θ on A as $f \;\Theta\; g$ if and only if $f = \Theta(g)$. Prove that Θ is an equivalence relation.

(b) Define a relation O on A as $f \;\mathrm{O}\; g$ if and only if $f = \mathrm{O}(g)$. Argue that O is not a partial order.

(c) Define a relation O on A/Θ as $[f]\;\mathrm{O}\;[g]$ if and only if $f = \mathrm{O}(g)$. Establish that this relation is well-defined. Prove that O is a partial order on A/Θ. Prove or disprove: O is a total order on A/Θ.

10 Let k be a fixed positive integer. Define a relation $\leqslant$ on $A = \mathbb{Z}^k$ as: $(a_1, a_2, \ldots, a_k) \leqslant (b_1, b_2, \ldots, b_k)$ if and only if $a_i \leqslant b_i$ for all $i = 1, 2, \ldots, k$. Prove that A is a lattice under this relation.