CS21001 Discrete Structures, Autumn 2007

Tutorial 4: August 16, 2007

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1 Let $A = \{1, 2, 3, 4, 5, 6\}$. Consider the relation

$$\rho = \{(1, 1), (1, 4), (1, 5), (2, 2), (2, 6), (3, 3), (4, 1), (4, 4), (4, 5), (5, 1), (5, 4), (5, 5), (6, 2), (6, 6)\}$$

on A. Convince yourself that ρ is an equivalence relation on A. Find the equivalence classes of all the elements of A. Explicitly write the quotient set A/ρ.

Solution The equivalence classes are $[1] = [4] = [5] = \{1, 4, 5\}$, $[2] = [6] = \{2, 6\}$ and $[3] = \{3\}$. Therefore, $A/\rho = \Big\{\{1, 4, 5\}, \{2, 6\}, \{3\}\Big\}$.

2 Let A be the set of all non-empty finite subsets of $\mathbb{Z}$. Define a relation ρ on A as: $U\ \rho\ V$ if and only if $\min(U) = \min(V)$. Also define the relation σ on A as: $U\ \sigma\ V$ if and only if $\min(U) \leqslant \min(V)$.

(a) Prove that ρ is an equivalence relation on A.

Solution For all $U, V, W \in A$ we have:
(1) $\min(U) = \min(U)$ [ρ is reflexive],
(2) $\min(U) = \min(V)$ implies $\min(V) = \min(U)$ [ρ is symmetric], and
(3) $\min(U) = \min(V)$ and $\min(V) = \min(W)$ imply $\min(U) = \min(W)$ [ρ is transitive].

(b) Identify good representatives from the equivalence classes of ρ.

Solution Consider the singleton sets $\{a\}$ for all $a \in \mathbb{Z}$.

(c) Define a bijection between the quotient set A/ρ and $\mathbb{Z}$.

Solution Take $f : A/\rho \to \mathbb{Z}$ as $[\{a\}]_\rho \mapsto a$. Argue that f is well-defined, injective and surjective.

(d) Prove or disprove: σ is a partial order on A.

Solution No, since the relation σ is not antisymmetric, i.e., $U\ \sigma\ V$ and $V\ \sigma\ U$ imply $\min(U) = \min(V)$, but we may have $U \neq V$ as in the case of $U = \{1, 2\}$ and $V = \{1, 3\}$, for example.

3 Let $f : A \to B$ be a function and σ an equivalence relation on B. Define a relation ρ on A as: $a\ \rho\ a'$ if and only if $f(a)\ \sigma\ f(a')$.

(a) Prove that ρ is an equivalence relation on A.

Solution Let $a, a', a'' \in A$.

[ρ is reflexive] Clearly, $f(a)\ \sigma\ f(a)$ (since σ is reflexive), i.e., $a\ \rho\ a$.

[ρ is symmetric] Also $a\ \rho\ a'$ implies $f(a)\ \sigma\ f(a')$, i.e., $f(a')\ \sigma\ f(a)$ (since σ is symmetric), i.e., $a'\ \rho\ a$.

[ρ is transitive] Finally, $a\ \rho\ a'$ and $a'\ \rho\ a''$ imply $f(a)\ \sigma\ f(a')$ and $f(a')\ \sigma\ f(a'')$, i.e., $f(a)\ \sigma\ f(a'')$ (since σ is transitive), i.e., $a\ \rho\ a''$.

(b) Define a map $\bar{f} : A/\rho \to B/\sigma$ as $[a]_\rho \mapsto [f(a)]_\sigma$. Prove that f is well-defined.

Solution Suppose $[a]_\rho = [a']_\rho$, i.e., $a\ \rho\ a'$, i.e., $f(a)\ \sigma\ f(a')$, i.e., $[f(a)]_\sigma = [f(a')]_\sigma$.

[The question of well-defined-ness arises here, because the value of the function is defined in terms of a representative of a class. Thus, we needed to show that irrespective of the choice of the representative, we get the same value for the function. The assignment $g : \mathbb{Z}_5 \to \mathbb{Z}_6$ taking $[a]_5 \mapsto [a]_6$ is not well-defined. For example, $[0]_5 = [5]_5$, but $[0]_6 \neq [5]_6$, i.e., we get different values when we use different representatives of the same class in the argument.]

(c) Prove that $\bar{f}$ is injective.

Solution Suppose $\bar{f}([a]_\rho) = \bar{f}([a']_\rho)$, i.e., $[f(a)]_\sigma = [f(a')]_\sigma$, i.e., $f(a)\ \sigma\ f(a')$, i.e., $a\ \rho\ a'$, i.e., $[a]_\rho = [a']_\rho$. So $\bar{f}$ is injective.

(d) Prove or disprove: If f is a bijection, then so also is $\bar{f}$.

Solution This is true. By Part (c), $\bar{f}$ is injective. On the other hand, take any $[b]_\sigma \in B/\sigma$. Since f is surjective, we have $b = f(a)$ for some $a \in A$. But then $\bar{f}([a]_\rho) = [f(a)]_\sigma = [b]_\sigma$, i.e., $\bar{f}$ is surjective too.

[Note that we never used the fact that f is injective. Indeed, $\bar{f}$ is bijective, whenever f is surjective.]

(e) Prove or disprove: If $\bar{f}$ is a bijection, then so also is f.

Solution This is false. Take $A = \{a, b, c\}$, $B = \{1, 2\}$ and $\sigma = \{(1, 1), (2, 2)\}$. Also define f as $f(a) = f(b) = 1$ and $f(c) = 2$. Then $\rho = \{(a, a), (b, b), (a, b), (b, a), (c, c)\}$, i.e., $A/\rho = \{\{a, b\}, \{c\}\}$, $B/\sigma = \{\{1\}, \{2\}\}$, and $\bar{f}(\{a, b\}) = \{1\}$ and $\bar{f}(\{c\}) = \{2\}$. Therefore, $\bar{f}$ is a bijection, whereas f is not.

Additional exercises

4 Let $A = \{1, 2, 3, 4, 5\}$. Define the relation ρ on A as follows.

$$\rho = \{(1, 1), (1, 2), (1, 3), (1, 5), (2, 1), (2, 4), (3, 3), (4, 5)\}.$$

(a) Determine the reflexive closure of ρ.
(b) Determine the symmetric closure of ρ.
(c) Determine the antisymmetric closure of ρ.
(d) Determine the transitive closure of ρ.

5 Define the relation ρ on $\mathbb{R}$ as: $a \; \rho \; b$ if and only if $a - b \in \mathbb{Z}$.

(a) Prove that ρ is an equivalence relation on $\mathbb{R}$.
(b) Find good representatives for the equivalence classes of ρ.
(c) Provide an explicit bijection between $\mathbb{R}/\rho$ and the real interval $(3, 5]$.

6 Let m, n be positive integers. Prove that the assignment $f : \mathbb{Z}_m \to \mathbb{Z}_n$ taking $[a]_m \mapsto [a]_n$ is well-defined if and only if m is an integral multiple of n.

7 Let $f : A \to B$ be a function, ρ an equivalence relation on A, and σ an equivalence relation on B. Suppose further that if $a \; \rho \; a'$, then $f(a) \; \sigma \; f(a')$. Define the map $\bar{f} : A/\rho \to B/\sigma$ as $\bar{f}([a]_\rho) = [f(a)]_\sigma$. Argue that $\bar{f}$ is well-defined. Prove or disprove the following statements.

(a) $\bar{f}$ is injective.
(b) If f is a bijection, then so also is $\bar{f}$.
(c) If $\bar{f}$ is a bijection, then so also is f.

8 Let A be the set of all non-empty finite subsets of $\mathbb{Z}$. Define a relation τ on A as: $U \; \tau \; V$ if and only if either $U = V$ or $\min(U) < \min(V)$. Prove or disprove: τ is a partial order on A.

9 Let A be the set of all functions $\mathbb{R} \to \mathbb{R}$. Define relations ρ, σ, τ on A as follows.

$$\begin{aligned} f \; \rho \; g &\quad \text{if and only if} \quad f(a) \leqslant g(a) \text{ for all } a \in \mathbb{R}, \\ f \; \sigma \; g &\quad \text{if and only if} \quad f(0) \leqslant g(0), \\ f \; \tau \; g &\quad \text{if and only if} \quad f(0) = g(0). \end{aligned}$$

Argue which of the relations ρ, σ, τ is/are equivalence relation(s). Argue which is/are partial order(s).

10 Prove the following assertions about a relation ρ on a set A.

(a) ρ is both symmetric and antisymmetric if and only if $\rho \subseteq \Delta_A$.
(b) ρ is transitive if and only if $\rho \circ \rho = \rho$.
(c) If ρ is non-empty, then ρ is an equivalence relation if and only if $\rho \circ \rho^{-1} = \rho$.
(d) ρ is a partial order if and only if ρ^{-1} is a partial order.