CS21001 Discrete Structures, Autumn 2007

Tutorial 3: August 09, 2007

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1 Let $f : A \to B$ be a function. Prove that f is a bijection if and only if there exists a function $g : B \to A$ with the properties that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

Solution [if] We first claim that f is injective. For the proof, take $a_1, a_2 \in A$ with $f(a_1) = f(a_2)$. Application of g yields $a_1 = g(f(a_1)) = g(f(a_2)) = a_2$. We then show that f is surjective. Take any $b \in B$. Call $a = g(b)$. But then $b = f(g(b)) = f(a)$, i.e., $b \in \operatorname{Im} f$.

[only if] If f is a bijection, $f^{-1} : B \to A$ is a total function which satisfies $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$.

2 Let $f : A \to B$ be a function, $S, S' \subseteq A$ and $T, T' \subseteq B$. Define

$$\begin{aligned} f(S) &= \{f(a) \mid a \in S\} \subseteq B, \\ f^{-1}(T) &= \{a \in A \mid f(a) \in T\} \subseteq A. \end{aligned}$$

Notice that f^{-1} is not necessarily a function from B to A. It maps subsets of B to subsets of A (and so can be treated as a function $\mathcal{P}(B) \to \mathcal{P}(A)$). However, if f is a bijection, then f^{-1} is naturally a function from B to A. If f is injective, then f^{-1} is a partial function not defined for the elements in $B \setminus f(A)$.

(a) If $S \subseteq S'$, then prove that $f(S) \subseteq f(S')$.

Solution Let $b \in f(S)$, i.e., $b = f(a)$ for some $a \in S$. Since $S \subseteq S'$, we have $a \in S'$ too and so $b = f(a) \in f(S')$.

(b) If $T \subseteq T'$, then prove that $f^{-1}(T) \subseteq f^{-1}(T')$.

Solution Let $a \in f^{-1}(T)$, i.e., $f(a) \in T$. Since $T \subseteq T'$, we have $f(a) \in T'$ too and so $a \in f^{-1}(T')$.

(c) Prove that $S \subseteq f^{-1}(f(S))$.

Solution Let $a \in S$. Then $f(a) \in f(S)$, i.e., $a \in f^{-1}(f(S))$.

(d) Give an example in which $S \subsetneqq f^{-1}(f(S))$.

Solution Take $A = B = \mathbb{Z}$, $f(n) = n^2$ and $S = \{1, 2, 3\}$. Then $f(S) = \{1, 4, 9\}$ and $f^{-1}(f(S)) = \{1, -1, 2, -2, 3, -3\}$ is a proper superset of S.

(e) Prove that $f(f^{-1}(T)) \subseteq T$.

Solution Let $b \in f(f^{-1}(T))$, i.e., $b = f(a)$ for some $a \in f^{-1}(T)$. But then $f(a) = b \in T$.

(f) Give an example in which $f(f^{-1}(T)) \subsetneqq T$.

Solution Take $A = B = \mathbb{N}$, $f(n) = n^2$ and $T = \{1, 2, 4\}$. We have $f^{-1}(T) = \{1, 2\}$ and so $f(f^{-1}(T)) = \{1, 4\}$ is a proper subset of T.

(g) Prove that $f(f^{-1}(f(S))) = f(S)$.

Solution By Parts (a) and (c), $f(S) \subseteq f(f^{-1}(f(S)))$. On the other hand, putting $T = f(S)$ in Part (e) yields $f(f^{-1}(f(S))) \subseteq f(S)$.

(h) Prove that $f^{-1}(f(f^{-1}(T))) = f^{-1}(T)$.

Solution By Parts (b) and (e), $f^{-1}(f(f^{-1}(T))) \subseteq f^{-1}(T)$. On the other hand, putting $S = f^{-1}(T)$ in Part (c) yields $f^{-1}(T) \subseteq f^{-1}(f(f^{-1}(T)))$.

A general comment: In order to prove that two sets X and Y are equal, it suffices to show $X \subseteq Y$ and $Y \subseteq X$. That is, choose an arbitrary element $x \in X$ and show that $x \in Y$ too. Moreover, take an arbitrary element $y \in Y$ and show that $y \in X$ too.

Additional exercises

3 Let $f : A \to B$ and $g : B \to C$ be functions.

(a) Prove that if the function $g \circ f : A \to C$ is injective, then f is injective.
(b) Give an example in which $g \circ f$ is injective but, g is not injective.
(c) Prove that if $g \circ f$ is surjective, then g is surjective.
(d) Give an example in which $g \circ f$ is surjective, but f is not surjective.

4 Let $f : A \to B$ be a function. Prove that

(a) f is injective if and only if $f^{-1}(f(S)) = S$ for every $S \subseteq A$.
(b) f is surjective if and only if $f(f^{-1}(T)) = T$ for every $T \subseteq B$.

5 Let $f : A \to B$ be a function, $S, S' \subseteq A$ and $T, T' \subseteq B$. Prove that

(a) $f(S \cup S') = f(S) \cup f(S')$.
(b) $f(S \cap S') = f(S) \cap f(S')$.
(c) $f^{-1}(T \cup T') = f^{-1}(T) \cup f^{-1}(T')$.
(d) $f^{-1}(T \cap T') = f^{-1}(T) \cap f^{-1}(T')$.
(e) $f^{-1}(\overline{T}) = \overline{f^{-1}(T)}$, where the complements in the left and right sides are in B and A, respectively.
(f) If f is surjective, then $f(\overline{S}) \supseteq \overline{f(S)}$, where the complements in the left and right sides are in A and B, respectively.

6 A function $f : \mathbb{Z} \to \mathbb{Z}$ is called *nilpotent* if for some $n \in \mathbb{N}$ we have $f^n(a) = 0$ for all $a \in \mathbb{Z}$.

(a) Give an example of a non-constant nilpotent function.
(b) Prove or disprove: The function $f(a) = \lfloor |a|/2 \rfloor$ is nilpotent.

7 Let $f : A \to B$ be a function. Define a function $\mathcal{F} : \mathcal{P}(A) \to \mathcal{P}(B)$ as $\mathcal{F}(S) = f(S)$ for all $S \subseteq A$. Prove that

(a) $\mathcal{F}$ is injective if and only if f is injective.
(b) $\mathcal{F}$ is surjective if and only if f is surjective.
(c) $\mathcal{F}$ is bijective if and only if f is bijective.

8 Let $f : A \to A$ be a function.

(a) Suppose that A is a finite set. Prove that f is injective if and only if f is surjective.
(b) Give an example of a function $f : A \to A$ which is injective but not surjective.
(c) Give an example of a function $f : A \to A$ which is surjective but not injective.

9 A function $f : \mathbb{R} \to \mathbb{R}$ is called *monotonic increasing* if $f(a) \leqslant f(b)$ whenever $a \leqslant b$. It is called *strictly monotonic increasing* if $f(a) < f(b)$ whenever $a < b$. One can define *monotonic decreasing* and *strictly monotonic decreasing* functions in analogous ways.

(a) Prove that a strictly monotonic increasing function is injective.
(b) Demonstrate than an injective function $\mathbb{R} \to \mathbb{R}$ need not be strictly increasing or strictly decreasing.
(c) Prove that a continuous injective function $\mathbb{R} \to \mathbb{R}$ is either strictly increasing or strictly decreasing.

10 Let $x, y \in \mathbb{R}$, $\lfloor\ \rfloor$ the floor function and $\lceil\ \rceil$ the ceiling function. Prove the following assertions.

(a) $x - 1 < \lfloor x \rfloor \leqslant x \leqslant \lceil x \rceil < x + 1$.
(b) $\lfloor -x \rfloor = -\lceil x \rceil$ and $\lceil -x \rceil = -\lfloor x \rfloor$.
(c) $\left\lfloor \left\lfloor \frac{x}{m} \right\rfloor / n \right\rfloor = \left\lfloor \frac{x}{mn} \right\rfloor$ and $\left\lceil \left\lceil \frac{x}{m} \right\rceil / n \right\rceil = \left\lceil \frac{x}{mn} \right\rceil$, where m, n are positive integers.
(d) $\lfloor mx \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{m} \right\rfloor + \left\lfloor x + \frac{2}{m} \right\rfloor + \cdots + \left\lfloor x + \frac{m-1}{m} \right\rfloor$, where m is a positive integer.