CS21001 Discrete Structures, Autumn 2007

Tutorial 2: August 02, 2007

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1 Let $P(x), Q(x)$ be predicates. Prove or disprove:

(a) $(\forall x)[P(x) \rightarrow Q(x)] \equiv (\forall x)[P(x)] \rightarrow (\forall x)[Q(x)]$.

Solution False. Let the universe of discourse be $D = \{a, b\}$. Take

$$P(a) = \mathrm{T}, \quad P(b) = \mathrm{F}, \qquad Q(a) = \mathrm{F}, \quad Q(b) = \mathrm{T}.$$

Then, the left side evaluates to F and the right side to T.

(b) $(\forall x)[P(x) \rightarrow Q(x)] \equiv (\exists x)[P(x)] \rightarrow (\forall x)[Q(x)]$.

Solution False. Let the universe of discourse be $D = \{a, b\}$. Take

$$P(a) = \mathrm{T}, \quad P(b) = \mathrm{F}, \qquad Q(a) = \mathrm{T}, \quad Q(b) = \mathrm{F}.$$

Then, the left side evaluates to T and the right side to F.

Notice that in order to prove that two expressions are *not* equivalent, it does *not* suffice to convert the two expressions to prenex normal forms and to show that the two converted expressions are (well, look) different. This is because it is not true that the prenex normal form of an expression is unique.

On the other hand, if you can convert two expressions to the same expression in the prenex normal form, then you can conclude that the two original expressions *are* equivalent.

2 What is the contrapositive of $(\forall x)[P(x)] \rightarrow (\exists x)[Q(x) \vee R(x)]$?

Solution The contrapositive of the given implication is

$$\neg\Big((\exists x)[Q(x) \vee R(x)]\Big) \rightarrow \neg\Big((\forall x)[P(x)]\Big) \equiv (\forall x)[\neg Q(x) \wedge \neg R(x)] \rightarrow (\exists x)[\neg P(x)].$$

The contrapositive of an implication is an implication. Do not perform formula manipulation so as to destroy the implication structure.

3 Find the negation of the following statement: "The sum of two odd integers is even."

Solution In this case, the universe of discourse is $\mathbb{Z}$. The given statement can be written as

$$\begin{aligned} & (\forall x)(\forall y)[\mathrm{odd}(x) \wedge \mathrm{odd}(y) \rightarrow \mathrm{even}(x+y)] \;\equiv\; (\forall x)(\forall y)[\neg(\mathrm{odd}(x) \wedge \mathrm{odd}(y)) \vee \mathrm{even}(x+y)] \\ \equiv\; & (\forall x)(\forall y)[\neg\mathrm{odd}(x) \vee \neg\mathrm{odd}(y) \vee \mathrm{even}(x+y)] \end{aligned}$$

with obvious meanings of the predicates. The negation of this statement is

$$(\exists x)(\exists y)[\mathrm{odd}(x) \wedge \mathrm{odd}(y) \wedge \neg\mathrm{even}(x+y)],$$

which can be rendered in English as "There exist two odd integers whose sum is not even".

4 Convert to prenex normal form the following expressions.

(a) $(\forall y)[P(y)] \rightarrow (\exists z)[Q(z)]$.

Solution We have

$$\begin{aligned} & (\forall y)[P(y)] \rightarrow (\exists z)[Q(z)] \;\equiv\; \neg(\forall y)[P(y)] \vee (\exists z)[Q(z)] \;\equiv\; (\exists y)[\neg P(y)] \vee (\exists z)[Q(z)] \\ \equiv\; & (\exists x)[\neg P(x) \vee Q(x)] \;\equiv\; (\exists x)[P(x) \rightarrow Q(x)]. \end{aligned}$$

It is of interest to study the equivalence $(\forall y)[P(y)] \rightarrow (\exists z)[Q(z)] \equiv (\exists x)[P(x) \rightarrow Q(x)]$ intuitively. Take the universe of discourse to be the set of all horses (in a given herd), $P(x)$ the predicate that "the horse x is black", and $Q(x)$ the predicate that "the horse x is tall". The left side of the equivalence says "if all horses (in the herd) are black, then there is at least one horse (in the same herd) that is tall". On the other hand, the right side says "there exists a horse whose 'blackness' implies its 'tallness'." Does these two notions sound equivalent intuitively? Perhaps, no! The idea is that you must handle the implication connective carefully. Recall that both the statements "if $2+2=5$, then I am your instructor" and "if $2+2=5$, then I am not your instructor" are true. Neither seems to make any intuitive sense. When such implications occur in compound statements, you may fail to have an intuitive understanding of an equivalence. But whatever you correctly deduce using equivalence formulas must be *logically* correct.

(b) $(\forall x)\Big[(\forall y)[P(x,y)] \rightarrow (\exists z)[Q(x,z)]\Big]$.

Solution Proceed as in Part (a). Each of the following four expressions in prenex normal form is equivalent to the given statement.

$$(\forall x)(\exists y)(\exists z)[\neg P(x,y) \vee Q(x,z)], \qquad (\forall x)(\exists y)(\exists z)[P(x,y) \rightarrow Q(x,z)],$$
$$(\forall x)(\exists y)[\neg P(x,y) \vee Q(x,y)], \qquad (\forall x)(\exists y)[P(x,y) \rightarrow Q(x,y)].$$

(c) $\Big((\forall x)[P(x)] \wedge (\exists x)[Q(x)]\Big) \vee (\forall x)[R(x)]$.

Solution The given statement is equivalent to

$$\begin{aligned} & (\forall x)(\exists y)[P(x) \wedge Q(y)] \vee (\forall x)[R(x)] \;\equiv\; (\forall x)\Big[(\exists y)[P(x) \wedge Q(y)]\Big] \vee (\forall x)[R(x)] \\ \equiv\; & (\forall x)(\forall z)\Big[(\exists y)[P(x) \wedge Q(y)] \vee R(z)\Big] \;\equiv\; (\forall x)(\forall z)\Big[(\exists y)[(P(x) \wedge Q(y)) \vee R(z)]\Big] \\ \equiv\; & (\forall x)(\forall z)(\exists y)[(P(x) \wedge Q(y)) \vee R(z)]. \end{aligned}$$

Additional exercises

5 Argue whether the statement $(\exists x)[P(x) \rightarrow Q(x)]$ is logically equivalent to $(\exists x)[P(x)] \rightarrow (\exists x)[Q(x)]$. Also argue whether $(\forall x)\Big[P(x) \rightarrow (\forall y)[Q(y)]\Big]$ is logically equivalent to $(\forall x)[P(x)] \rightarrow (\forall y)[Q(y)]$.

6 Consider the following predicates.

$A(x)$	:	Program x implements the correct algorithm.
$B(x)$	:	Program x has bugs.
$C(x,y)$	:	Program x gives correct output upon input y.
$D(x,y)$	:	Program x halts upon input y.

Express the following general statement about a program: "If any program that gives correct outputs on all possible inputs implements the correct algorithm, then for some input any buggy program either does not halt or gives incorrect output." Write in English the negation, the contrapositive and the converse of the above statement.

7 Convert the following statements to prenex normal forms.

(a) $(\exists! x)P(x)$.

(b) $(\forall x)[P(x)] \wedge \Big((\exists x)[Q(x)] \vee (\forall x)[R(x)]\Big)$.

(c) $\Big((\forall x)[P(x)] \rightarrow (\forall x)[Q(x)]\Big) \rightarrow (\exists x)[R(x)]$.

(d) $(\forall x)[P(x)] \oplus (\exists x)[Q(x)]$.

(e) $(\forall x)[P(x)] \leftrightarrow (\exists x)[Q(x)]$.