CS21001 Discrete Structures, Autumn 2007

Tutorial 1: July 26, 2007

1 Let p, q, r be propositions. Prove that $((p \to q) \wedge (q \to r)) \to (p \to r)$ is a tautology.

Method 1: Using truth table

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$((p \to q) \wedge (q \to r)) \to (p \to r)$
F	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	T	T	T	T	T	T	T
T	F	F	F	T	F	F	T
T	F	T	F	T	F	T	T
T	T	F	T	F	F	F	T
T	T	T	T	T	T	T	T

The last column implies that $((p \to q) \wedge (q \to r)) \to (p \to r)$ is a tautology.

Method 2: Using equivalence formulas

We have $p \to q \equiv \neg p \vee q$ and $q \to r \equiv \neg q \vee r$, and so

$$\begin{aligned}
(p \to q) \wedge (q \to r) &\equiv (\neg p \vee q) \wedge (\neg q \vee r) \equiv [(\neg p \vee q) \wedge \neg q] \vee [(\neg p \vee q) \wedge r] \\
&\equiv (\neg p \wedge \neg q) \vee (q \wedge \neg q) \vee (\neg p \wedge r) \vee (q \wedge r) \\
&\equiv (\neg p \wedge \neg q) \vee \mathrm{F} \vee (\neg p \wedge r) \vee (q \wedge r) \\
&\equiv (\neg p \wedge \neg q) \vee (\neg p \wedge r) \vee (q \wedge r).
\end{aligned}$$

Therefore,

$$\begin{aligned}
((p \to q) \wedge (q \to r)) \to (p \to r) &\equiv \neg[(\neg p \wedge \neg q) \vee (\neg p \wedge r) \vee (q \wedge r)] \vee (\neg p \vee r) \\
&\equiv [(p \vee q) \wedge (p \vee \neg r) \wedge (\neg q \vee \neg r)] \vee (\neg p \vee r) \\
&\equiv (p \vee q \vee \neg p \vee r) \wedge (p \vee \neg r \vee \neg p \vee r) \wedge (\neg q \vee \neg r \vee \neg p \vee r) \\
&\equiv \mathrm{T} \wedge \mathrm{T} \wedge \mathrm{T} \equiv \mathrm{T}.
\end{aligned}$$

2 Let p, q, r be propositions. Prove or disprove: $(p \to q) \wedge (q \to r)$ is logically equivalent to $p \to r$.

Solution No, since the last two columns of the truth table in Exercise 1 are not identical.

3 Let p, q, r stand for the following propositions:

p: It is raining.
q: I have a headache.
r: I attend the lecture.

The proposition $P = (\neg p \wedge \neg q) \to r$ is rendered in English as follows: “If it is not raining and I do not have a headache, then I attend the lecture.”

(a) Write in English the negation of P.

Solution $P \equiv \neg(\neg p \wedge \neg q) \vee r \equiv (p \vee q \vee r)$, and so $\neg P \equiv (\neg p \wedge \neg q \wedge \neg r)$. This may be rendered in English as: “It is not raining and I do not have a headache and I do not attend the lecture.” In order to carry better sense, the second “and” may be replaced by “but” or even by “but still”. Notice that “but” in this usage is the same as “and”.

(b) Write in English the contrapositive of P.

Solution The contrapositive of P is $\neg r \to \neg(\neg p \wedge \neg q)$ and may be rendered in English as: "If I do not attend the lecture, then it is not that it is not raining and I do not have a headache." This is bad English. We can simplify $\neg(\neg p \wedge \neg q)$ as $p \vee q$, and so a better rephrasing is: "If I do not attend the lecture, then (either) it is raining or I have a headache (or both)."

Note that the contrapositive of an implication is another implication and should be presented as an implication. You may, however, simplify the two sides of the implication. The same applies to converses and inverses.

(c) Write in English the converse of P.

Solution The converse of P is $r \to (\neg p \wedge \neg q)$ and is rendered in English as: "If I attend the lecture, then it is not raining and I do not have a headache."

(d) Write in English the inverse of P.

Solution The inverse of P is $\neg(\neg p \wedge \neg q) \to \neg r$, i.e., $(p \vee q) \to \neg r$. In English this is: "If it is raining or I have a headache, then I do not attend the lecture."

Additional exercises

4 Let p, q, r be propositions. Which of the following statements are true?

(a) $(p \wedge q) \to (p \vee q)$ is a tautology.
(b) $(p \wedge q) \to (p \oplus q)$ is a tautology.
(c) $(p \vee q) \to (p \wedge q)$ is logically equivalent to $p \leftrightarrow q$.
(d) $(p \oplus q) \to (p \wedge q)$ is logically equivalent to $p \leftrightarrow q$.
(e) $(p \to q) \wedge (q \to r)$ is logically equivalent to $(p \to q) \wedge (p \to r)$.
(f) $(p \to q) \to r$ is logically equivalent to $(p \wedge q) \to r$.
(g) $\neg p \to (q \to r)$ is logically equivalent to $q \to (p \vee r)$.
(h) $p \to (q \vee r)$ is logically equivalent to $(p \wedge \neg q) \to r$.
(i) $\neg p \to (q \vee r)$ is logically equivalent to $\neg q \to (p \vee r)$.
(j) $(p \downarrow q) \downarrow r$ is logically equivalent to $p \downarrow (q \downarrow r)$.

5 Let p, q, r, s, t stand for the following propositions.

p: The program terminates.
q: The program gives correct output.
r: The program is syntactically correct.
s: The program has bugs.
t: The correct algorithm is implemented.

Consider the statement P: "If the program terminates, but gives incorrect output, then the program is syntactically correct, but either the program has bugs or the correct algorithm is not implemented."

(a) Write P as a logical proposition involving p, q, r, s, t.
(b) Write in English the negation of P.
(c) Write in English the contrapositive of P.
(d) Write in English the converse of P.
(e) Write in English the inverse of P.
(f) Write the following statement as a logical proposition: "Although the program has bugs, it terminates and gives the correct output."
(g) Write the following statement as a logical proposition: "Since the program has bugs, it either does not terminate or does not give the correct output."

6 Express $(p \oplus q) \downarrow r$ in CNF and DNF, where p, q, r are atomic propositions.

7 Prove that $\{\oplus, \wedge\}$ is a functionally complete set of connectives.