CS21001 Discrete Structures, Autumn 2007

Mid-semester examination

Total marks: 60 | September 2007 | **Duration:** 2 hours

[*Answer all questions. Be brief and precise.*]

1 Consider the following recursive C function.

```
unsigned int f ( unsigned int n )
{
   if ((n == 0) || (n == 1)) return 0;
   if ((n%2) == 0) return 1 + f(n/2);
   return 1 + f(5*n+1);
}
```

(a) What does $f(19)$ return? **(5)**

(b) What does $f(5)$ return? **(5)**

(c) What can you conclude about f as a function $\mathbb{N} \to \mathbb{N}$? **(5)**

2 Let $\mathbb{C}$ denote the set of complex numbers and $\mathbb{Z}[\mathrm{i}]$ the subset $\{a + \mathrm{i}b \mid a, b \in \mathbb{Z}\}$ of $\mathbb{C}$. Elements of $\mathbb{Z}[\mathrm{i}]$ are called *Gaussian integers*. For $z = x + \mathrm{i}y \in \mathbb{C}$, we denote by $|z|$ the magnitude of z and by $\arg z$ the argument of z. Thus, $z = \sqrt{x^2 + y^2}$ and $\arg z = \tan^{-1}\left(\frac{y}{x}\right)$. We take $\arg z$ in the interval $[0, 2\pi)$.

Define a relation ρ on $\mathbb{C}$ as follows. Take $z_1, z_2 \in \mathbb{C}$. We say that $z_1 \; \rho \; z_2$ if and only if

either (i) $|z_1| < |z_2|$,
or (ii) $|z_1| = |z_2|$ and $\arg z_1 \leqslant \arg z_2$.

Also define a relation σ on $\mathbb{C}$ as $z_1 \; \sigma \; z_2$ if and only if $|z_1| = |z_2|$.

(a) Prove that ρ is a partial order on $\mathbb{C}$. **(5)**

(b) Prove that ρ is a well-ordering of $\mathbb{Z}[\mathrm{i}]$. **(5)**

(c) Prove that σ is an equivalence relation on $\mathbb{C}$. **(5)**

(d) What are the equivalence classes of σ? (Provide a geometric description.) **(5)**

3 For real numbers a, b with $a < b$, we define the *closed interval* $[a, b] = \{x \in \mathbb{R} \mid a \leqslant x \leqslant b\}$ and the *open interval* $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$.

(a) Prove that the closed interval $[0, 1]$ is equinumerous with the open interval $(0, 1)$. **(5)**

(b) Provide an explicit bijection between $\mathbb{R}$ and $\mathbb{R} \setminus \{0\}$. **(5)**

4 **(a)** Use a diagonalization argument to prove that the set of all infinite sequences of natural numbers is uncountable. **(5)**

(b) Conclude that the set of all functions $\mathbb{N} \to \mathbb{N}$ is uncountable. **(5)**

(c) Prove that the set of all finite sequences of natural numbers is countable. **(5)**

Dr. Abhijit Das, Dept. of Computer Science & Engineering, IIT Kharagpur, India