CS21001 Discrete Structures, Autumn 2007

Mid-semester examination

Total marks: 60 September 2007 **Duration:** 2 hours

[*Answer all questions. Be brief and precise.*]

1 Consider the following recursive C function.

```
unsigned int f ( unsigned int n )
{
   if ((n == 0) || (n == 1)) return 0;
   if ((n%2) == 0) return 1 + f(n/2);
   return 1 + f(5*n+1);
}
```

(a) What does $f(19)$ return? **(5)**

Solution We have $f(19) = 1 + f(96) = 2 + f(48) = 3 + f(24) = 4 + f(12) = 5 + f(6) = 6 + f(3) = 7 + f(16) = 8 + f(8) = 9 + f(4) = 10 + f(2) = 11 + f(1) = 11 + 0 = 11$.

(b) What does $f(5)$ return? **(5)**

Solution We have $f(5) = 1 + f(26) = 2 + f(13) = 3 + f(66) = 4 + f(33) = 5 + f(166) = 6 + f(83) = 7 + f(416) = 8 + f(208) = 9 + f(104) = 10 + f(52) = 11 + f(26) = 12 + f(13) = \cdots = 22 + f(13) = \cdots = 32 + f(13) = \cdots = 42 + f(13) = \cdots$. Thus the above function does not terminate when 5 is passed as its argument. When the recursion stack runs out of memory, it exits with an error message (typically `segmentation fault`).

(c) What can you conclude about f as a function $\mathbb{N} \to \mathbb{N}$? **(5)**

Solution The sequence of computation in Part (b) implies that $f(13) = 10 + f(13)$, i.e., f is not well-defined as a function $\mathbb{N} \to \mathbb{N}$.

2 Let $\mathbb{C}$ denote the set of complex numbers and $\mathbb{Z}[\mathrm{i}]$ the subset $\{a + \mathrm{i}b \mid a, b \in \mathbb{Z}\}$ of $\mathbb{C}$. Elements of $\mathbb{Z}[\mathrm{i}]$ are called *Gaussian integers*. For $z = x + \mathrm{i}y \in \mathbb{C}$, we denote by $|z|$ the magnitude of z and by $\arg z$ the argument of z. Thus, $z = \sqrt{x^2 + y^2}$ and $\arg z = \tan^{-1}\left(\frac{y}{x}\right)$. We take $\arg z$ in the interval $[0, 2\pi)$.
Define a relation ρ on $\mathbb{C}$ as follows. Take $z_1, z_2 \in \mathbb{C}$. We say that $z_1 \ \rho \ z_2$ if and only if

either (i) $|z_1| < |z_2|$,
or (ii) $|z_1| = |z_2|$ and $\arg z_1 \leqslant \arg z_2$.

Also define a relation σ on $\mathbb{C}$ as $z_1 \ \sigma \ z_2$ if and only if $|z_1| = |z_2|$.

(a) Prove that ρ is a partial order on $\mathbb{C}$. **(5)**

Solution Let $z, z_1, z_2, z_3 \in \mathbb{C}$. We have $|z| = |z|$ and $\arg z \leqslant \arg z$, i.e., $z \ \rho \ z$, i.e., ρ is reflexive. Then suppose $z_1 \ \rho \ z_2$ and $z_2 \ \rho \ z_1$. If $|z_1| < |z_2|$, we cannot have $z_2 \ \rho \ z_1$. Analogously, if $|z_2| < |z_1|$, we cannot have $z_1 \ \rho \ z_2$. Therefore, $|z_1| = |z_2|$. In that case, $\arg z_1 \leqslant \arg z_2$ and $\arg z_2 \leqslant \arg z_1$, i.e., $\arg z_1 = \arg z_2$. It follows that $z_1 = z_2$, i.e., ρ is antisymmetric. Finally, let $z_1 \ \rho \ z_2$ and $z_2 \ \rho \ z_3$. This means $|z_1| \leqslant |z_2| \leqslant |z_3|$. If $|z_1| < |z_2|$ or $|z_2| < |z_3|$, then $|z_1| < |z_3|$, i.e., $z_1 \ \rho \ z_3$. If $|z_1| = |z_2| = |z_3|$, we have $\arg z_1 \leqslant \arg z_2 \leqslant \arg z_3$, i.e., again $z_1 \ \rho \ z_3$. Thus, ρ is transitive.

(b) Prove that ρ is a well-ordering of $\mathbb{Z}[\mathrm{i}]$. **(5)**

Solution Let S be a non-empty subset of $\mathbb{Z}[\mathrm{i}]$. Consider the set $X = \{|z|^2 \mid z \in S\}$. X, being a non-empty subset of $\mathbb{N}$, contains a minimum element; call it n. Let $Y = \{z \in S \mid |z|^2 = n\}$. Since the equation $x^2 + y^2 = n$ can have only finitely many solutions in integer values of x and y, the set Y is finite. It is also non-empty. Thus, Y contains a minimum element; call it z. It is clear that this z is the minimum element of S with respect to ρ.

(c) Prove that σ is an equivalence relation on $\mathbb{C}$. **(5)**

Solution Let $z, z_1, z_2, z_3 \in \mathbb{C}$. Since $|z| = |z|$, we have $z \; \sigma \; z$, i.e., σ is reflexive. Also $z_1 \; \sigma \; z_2$ implies $|z_1| = |z_2|$, i.e., $|z_2| = |z_1|$, i.e., $z_2 \; \sigma \; z_1$, i.e., σ is symmetric. Finally, $z_1 \; \sigma \; z_2$ and $z_2 \; \sigma \; z_3$ imply $|z_1| = |z_2| = |z_3|$, i.e., $z_1 \; \sigma \; z_3$, i.e., σ is transitive too.

(d) What are the equivalence classes of σ? (Provide a geometric description.) **(5)**

Solution Let $z = x + \mathrm{i}y \in \mathbb{C}$ with $r = \sqrt{x^2 + y^2}$. Then $[z]_\sigma$ consists precisely of all complex numbers whose absolute values equal r, i.e., $[z]_\sigma$ is the circle of radius r centered at the origin.

3 For real numbers a, b with $a < b$, we define the *closed interval* $[a, b] = \{x \in \mathbb{R} \mid a \leqslant x \leqslant b\}$ and the *open interval* $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$.

(a) Prove that the closed interval $[0, 1]$ is equinumerous with the open interval $(0, 1)$. **(5)**

Solution The inclusion map $f : (0, 1) \to [0, 1]$ taking $x \mapsto x$ is injective. Also the map $g : [0, 1] \to (0, 1)$ taking $x \mapsto \frac{1}{4} + \frac{x}{2}$ is an (injective) embedding of $[0, 1]$ in the interval $[\frac{1}{4}, \frac{3}{4}]$ which is a subset of $(0, 1)$.

(b) Provide an explicit bijection between $\mathbb{R}$ and $\mathbb{R} \setminus \{0\}$. **(5)**

Solution For $n \in \mathbb{N}$, denote $I_n = [n, n + 1)$ and $J_n = (n, n + 1]$. For any fixed n, the map $f_n : I_n \to J_n$ taking $x \mapsto (2n + 1) - x$ is a bijection. We have the disjoint unions $\mathbb{R} = (-\infty, 0) \cup \left(\bigcup_{n \in \mathbb{N}} I_n\right)$ and $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup \left(\bigcup_{n \in \mathbb{N}} J_n\right)$. The map that relocates I_n to J_n using f_n for all $n \in \mathbb{N}$ and that fixes $(-\infty, 0)$ element-wise is a bijection $\mathbb{R} \to \mathbb{R} \setminus \{0\}$.

4 **(a)** Use a diagonalization argument to prove that the set of all infinite sequences of natural numbers is uncountable. **(5)**

Solution Let A be the set of all infinite sequences of natural numbers. Suppose that A is countable. Then there exists a bijective map $f : \mathbb{N} \to A$. Denote by $f(n)$ the sequence $a_{n0}, a_{n1}, a_{n2}, \ldots$. Define an infinite sequence $b_0, b_1, b_2, \ldots, b_n, \ldots$ of natural numbers as follows:

$$b_n = \begin{cases} 2 & \text{if } a_{nn} = 1, \\ 1 & \text{if } a_{nn} \neq 1. \end{cases}$$

Since f is bijective, the sequence $b_0, b_1, b_2, \ldots$ is equal to $f(n)$ for some $n \in \mathbb{N}$. But $b_n \neq a_{nn}$ by construction, i.e., the sequence $b_0, b_1, b_2, \ldots$ is different from $f(n)$, a contradiction.

(b) Conclude that the set of all functions $\mathbb{N} \to \mathbb{N}$ is uncountable. **(5)**

Solution Every sequence $a_0, a_1, a_2, \ldots, a_n, \ldots$ of natural numbers can be viewed as the unique function $\mathbb{N} \to \mathbb{N}$ taking $n \mapsto a_n$.

(c) Prove that the set of all finite sequences of natural numbers is countable. **(5)**

Solution Let C be the set of all finite sequences of natural numbers. We have $C = \bigcup_{n \in \mathbb{N}} C_n$, where C_n is the set of all sequences of natural numbers of length n. Since C_n can be viewed as the set $\mathbb{N}^n$ of all (ordered) n-tuples of natural numbers and since $\mathbb{N}^n$ is countable for every $n \in \mathbb{N}$, each C_n is countable. Therefore, C is the union of countably many countable sets and so is countable too.