CS21001 Discrete Structures, Autumn 2007

Class test 1

Total marks: 50 September 06, 2007 (6:00-7:00pm) **Duration:** 1 hour

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Roll No: ______________ **Name:** ____________________________________

[*Write your answers in the respective spaces provided in the question paper itself. Answer briefly.*]

1 Prove or disprove the following assertions. **(5 $\times$ 5)**

(a) Let p, q, r, s be atomic propositions. Then, the compound proposition $p \wedge r \rightarrow q \vee \neg s$ is logically equivalent to the compound proposition $p \wedge s \rightarrow q \vee \neg r$.

Solution True. $p \wedge r \rightarrow q \vee \neg s \equiv \neg(p \wedge r) \vee (q \vee \neg s) \equiv \neg p \vee \neg r \vee q \vee \neg s \equiv (\neg p \vee \neg s) \vee (q \vee \neg r) \equiv \neg(p \wedge s) \vee (q \vee \neg r) \equiv p \wedge s \rightarrow q \vee \neg r$.

(b) Let $P(x), Q(x)$ be predicates. Then the proposition $(\exists x)[P(x)] \rightarrow (\exists y)[Q(y)]$ is logically equivalent to $(\exists x)(\exists y)[P(x) \rightarrow Q(y)]$.

Solution False. Let the universe of discourse be $\{1, 2\}$. Take $P(1) =$ T, $P(2) =$ F, $Q(1) = Q(2) =$ F. Then $(\exists x)[P(x)] \rightarrow (\exists y)[Q(y)]$ evaluates to false, whereas $(\exists x)(\exists y)[P(x) \rightarrow Q(y)]$ evaluates to true.

(c) Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be functions such that $g \circ f : A \rightarrow C$ is a bijection. Then f must be a bijection too.

Solution False. Let $A = C = \{1, 2\}$, $B = \{1, 2, 3\}$. Take $f(1) = 1$, $f(2) = 2$, $g(1) = 1$, and $g(2) = g(3) = 2$. Then $g \circ f$ is a bijection (the identity map), whereas f is clearly not surjective.

(d) The antisymmetric closure of a relation ρ on a set A exists if and only if ρ itself is antisymmetric.

Solution True. If ρ is already antisymmetric, then the antisymmetric closure of ρ is ρ itself. On the other hand, suppose that ρ is not antisymmetric. This means that ρ contains two pairs (a, b) and (b, a) for some $a \neq b$. But then any superset of ρ continues to contain these pairs and, therefore, cannot be antisymmetric.

(e) There exists a relation that well-orders $\mathbb{Z}$.

Solution True. Consider the relation $\preceq$ on $\mathbb{Z}$ defined as $0 \preceq 1 \preceq -1 \preceq 2 \preceq -2 \preceq 3 \preceq -3 \preceq \cdots$. It is easy to argue that $\preceq$ well-orders $\mathbb{Z}$.

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2 Let F_n, $n \in \mathbb{N}$, denote the sequence of Fibonacci numbers. Prove by induction on n that **(15)**

$$F_{2n} = F_n(F_{n+1} + F_{n-1}) \quad \text{and} \quad F_{2n+1} = F_{n+1}^2 + F_n^2 \quad \text{for all } \ n \geqslant 1.$$

Solution [Basis] Take $n = 1$. In this case, $F_{2n} = F_2 = 1$, whereas $F_n(F_{n+1} + F_{n-1}) = F_1(F_2 + F_0) = 1 \times (1 + 0) = 1$. Moreover, $F_{2n+1} = F_3 = 2 = 1^2 + 1^2 = F_{n+1}^2 + F_n^2$.

[Induction] Suppose that $F_{2n} = F_n(F_{n+1} + F_{n-1})$ and $F_{2n+1} = F_{n+1}^2 + F_n^2$ for some $n \geqslant 1$. We have $F_{2(n+1)} = F_{2n+2} = F_{2n+1} + F_{2n} = (F_{n+1}^2 + F_n^2) + F_n(F_{n+1} + F_{n-1}) = F_{n+1}(F_{n+1} + F_n) + F_n(F_n + F_{n-1}) = F_{n+1}F_{n+2} + F_nF_{n+1} = F_{n+1}(F_{n+2} + F_n)$. Moreover, $F_{2(n+1)+1} = F_{2n+3} = F_{2n+2} + F_{2n+1} = F_{n+1}(F_{n+2} + F_n) + (F_{n+1}^2 + F_n^2) = F_{n+1}(F_{n+1} + 2F_n) + (F_{n+1}^2 + F_n^2) = (F_{n+1}^2 + 2F_{n+1}F_n + F_n^2) + F_{n+1}^2 = (F_{n+1} + F_n)^2 + F_{n+1}^2 = F_{n+2}^2 + F_{n+1}^2$.

3 Let $f : \mathbb{N} \to \mathbb{N}$ be a bijection not equal to the identity map. Prove that there exists $n \in \mathbb{N}$ such that $n < f(n)$ and $n < f^{-1}(n)$. **(10)**

Solution Let $S = \{a \in \mathbb{N} \mid f(a) \neq a\}$. Since f is not the identity map, we have $S \neq \emptyset$. Let n be the minimum element in S. Thus, $f(0) = 0, f(1) = 1, \ldots, f(n-1) = n-1$. Since f is injective, $f(n)$ cannot be equal to $0, 1, 2, \ldots, n-1$. Moreover, since $f(n) \neq n$, we must have $f(n) > n$. Moreover, $f^{-1}(0) = 0, f^{-1}(1) = 1, \ldots, f^{-1}(n-1) = n-1$, whereas $f^{-1}(n) \neq n$ (since $f(n) > n$ and f is injective). Therefore, $f^{-1}(n) > n$ too.