CS21001 Discrete Structures, Autumn 2006

Mid-semester examination

Total marks: 100 | September 15, 2006 (AN): S-302 (B) | **Duration:** 2 hours

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[*Answer* <u>*all*</u> *questions*]

1. Let p, q be propositions, and $P(x), Q(x), R(x)$ be predicates.

(a) Prove that $(p \vee q) \Rightarrow (p \wedge q)$ is logically equivalent to $p \Leftrightarrow q$. **(5)**

(b) Prove that $(p \wedge q) \Rightarrow (p \vee q)$ is a tautology. **(5)**

(c) What is the contrapositive of $\forall x\, [P(x)] \Rightarrow \exists x\, [Q(x) \vee R(x)]$? **(5)**

(d) Write in English the converse of the following assertion:
"If I wake up early, I attend the lecture, unless I have a headache." **(5)**

(e) Write in English the negation of the following assertion:
"The sum of any two odd integers is an even integer." **(5)**

2. Let a_n be the value returned by the following C function upon input n. (Assume that C supports arbitrarily big integers, so that no overflow occurs during arithmetic operations.)

```
unsigned int a ( unsigned int n )
{
   unsigned int sum, i;

   if (n == 0) return 1;
   sum = 0;
   for (i=0; i<n; ++i) sum += a(i);
   return sum;
}
```

(a) Derive a recurrence relation for the sequence $a_n, n \geqslant 0$. **(10)**

(b) Prove by induction on n that $a_n = 2^{n-1}$ for all $n \geqslant 1$. **(10)**

3. For $n \geqslant 0$ let A_n denote the set $\{0, 1, 2, \ldots, 3^n - 1\}$. Moreover, let B_n be the set of those elements of A_n whose ternary representations (representations in base 3) contain 1, and C_n the set of those elements of A_n whose ternary representations do not contain 1. The ternary representations of some small integers are:

$$
\begin{aligned}
0 &= (0)_3 = (00)_3 = (000)_3 = (0000)_3 = \cdots \\
1 &= (1)_3 = (01)_3 = (001)_3 = (0001)_3 = \cdots \\
2 &= (2)_3 = (02)_3 = (002)_3 = (0002)_3 = \cdots \\
3 &= (10)_3 = (010)_3 = (0010)_3 = \cdots \\
4 &= (11)_3 = (011)_3 = (0011)_3 = \cdots \\
5 &= (12)_3 = (012)_3 = (0012)_3 = \cdots \\
6 &= (20)_3 = (020)_3 = (0020)_3 = \cdots \\
7 &= (21)_3 = (021)_3 = (0021)_3 = \cdots \\
8 &= (22)_3 = (022)_3 = (0022)_3 = \cdots \\
9 &= (100)_3 = (0100)_3 = \cdots \\
10 &= (101)_3 = (0101)_3 = \cdots
\end{aligned}
$$

It follows that:

$$\begin{array}{lll} A_0 = \{0\}, & B_0 = \emptyset, & C_0 = \{0\}. \\ A_1 = \{0,1,2\}, & B_1 = \{1\}, & C_1 = \{0,2\}. \\ A_2 = \{0,1,2,3,4,5,6,7,8\}, & B_2 = \{1,3,4,5,7\}, & C_2 = \{0,2,6,8\}. \end{array}$$

Let r_n denote the size of C_n, s_n the sum of elements of B_n, and t_n the sum of elements of C_n. For example, $r_2 = 4$, $s_2 = 1+3+4+5+7 = 20$, and $t_2 = 0+2+6+8 = 16$.

(a) Prove that $r_n = 2^n$ for all $n \geqslant 0$. **(5)**

(b) Deduce that t_n satisfies the recurrence relation: **(10)**

$$\begin{array}{rcl} t_0 & = & 0, \\ t_n & = & 2t_{n-1} + \dfrac{1}{3} \times 6^n \text{ for all } n \geqslant 1. \end{array}$$

(c) Solve the above recurrence relation in order to derive that $t_n = \frac{1}{2}(6^n - 2^n)$ for all $n \geqslant 0$. **(10)**

(d) Conclude that $s_n = \frac{1}{2}(9^n - 6^n - 3^n + 2^n)$ for all $n \geqslant 0$. **(5)**

4. Let $k \in \mathbb{N}$, $S = \{1, 2, \ldots, k\}$, and $A = \mathcal{P}(S) \setminus \{\emptyset\}$, where $\mathcal{P}(S)$ denotes the power set of S, and $\emptyset$ denotes the empty set. In other words, the set A comprises all non-empty subsets of $\{1, 2, \ldots, k\}$. For each $a \in A$ denote by $\min(a)$ the smallest element of a (notice that here a is a set).

(a) Define a relation ρ on A as follows: $a \;\rho\; b$ if and only if $\min(a) = \min(b)$. Prove that ρ is an equivalence relation on A. **(5)**

(b) What is the size of the quotient set A/ρ? **(5)**

(c) Define a relation σ on A as follows: $a \;\sigma\; b$ if and only if either $a = b$ or $\min(a) < \min(b)$. Prove that σ is a partial order on A. **(5)**

(d) Is σ also a total order on A? **(5)**

(e) What is the total number of antisymmetric relations on a finite set of size n? **(5)**