CS21001 Discrete Structures, Autumn 2006

Mid-semester examination: Solutions

Total marks: 100 September 15, 2006 (AN): S-302 (B) **Duration:** 2 hours

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1. Let p, q be propositions, and $P(x), Q(x), R(x)$ be predicates.

(a) Prove that $(p \vee q) \Rightarrow (p \wedge q)$ is logically equivalent to $p \Leftrightarrow q$.

Solution $(p \vee q) \Rightarrow (p \wedge q) \equiv \neg(p \vee q) \vee (p \wedge q) \equiv (\neg p \wedge \neg q) \vee (p \wedge q) \equiv p \Leftrightarrow q$.

(b) Prove that $(p \wedge q) \Rightarrow (p \vee q)$ is a tautology.

Solution $(p \wedge q) \Rightarrow (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q) \equiv (\neg p \vee \neg q) \vee (p \vee q) \equiv (\neg p \vee p) \vee (\neg q \vee q) \equiv$ True $\vee$ True $\equiv$ True.

(c) What is the contraposition of $\forall x\,[P(x)] \Rightarrow \exists x\,[Q(x) \vee R(x)]$?

Solution $\forall x\,[\neg Q(x) \wedge \neg R(x)] \Rightarrow \exists x\,[\neg P(x)]$.

(d) Write in English the converse of the following assertion: "If I wake up early, I attend the lecture, unless I have a headache."

Solution "If I attend the lecture, I wake (have woken) up early and do not have a headache."

(e) Write in English the negation of the following assertion: "The sum of any two odd integers is an even integer."

Solution "There exist two odd integers the sum of which is an odd integer."

2. Let a_n be the value returned by the following C function upon input n.

```
unsigned int a ( unsigned int n )
{
   unsigned int sum, i;

   if (n == 0) return 1;
   sum = 0;
   for (i=0; i<n; ++i) sum += a(i);
   return sum;
}
```

(a) Derive a recurrence relation for the sequence $a_n, n \geqslant 0$.

Solution

$$\begin{aligned} a_0 &= 1, \\ a_n &= a_{n-1} + a_{n-2} + \cdots + a_1 + a_0 \;\text{ for }\; n \geqslant 1. \end{aligned}$$

(b) Prove by induction on n that $a_n = 2^{n-1}$ for all $n \geqslant 1$.

Solution [Basis] For $n = 1$ we have $a_1 = a_0 = 1 = 2^{1-1}$.

[Induction] Suppose $a_i = 2^{i-1}$ for $i = 1, 2, \ldots, n$. But then $a_{n+1} = (a_n + a_{n-1} + \cdots + a_2 + a_1) + a_0 = (2^{n-1} + 2^{n-2} + \cdots + 2^1 + 2^0) + 1 = (2^n - 1) + 1 = 2^n = 2^{(n+1)-1}$.

3. For $n \geqslant 0$ let A_n denote the set $\{0, 1, 2, \ldots, 3^n - 1\}$. Moreover, let B_n be the set of those elements of A_n whose ternary representations (representations in base 3) contain 1, and C_n the set of those elements of A_n whose ternary representations do not contain 1. The ternary representations of some small integers are:

$$\begin{aligned}
0 &= (0)_3 = (00)_3 = (000)_3 = (0000)_3 = \cdots \\
1 &= (1)_3 = (01)_3 = (001)_3 = (0001)_3 = \cdots \\
2 &= (2)_3 = (02)_3 = (002)_3 = (0002)_3 = \cdots \\
3 &= (10)_3 = (010)_3 = (0010)_3 = \cdots \\
4 &= (11)_3 = (011)_3 = (0011)_3 = \cdots \\
5 &= (12)_3 = (012)_3 = (0012)_3 = \cdots \\
6 &= (20)_3 = (020)_3 = (0020)_3 = \cdots \\
7 &= (21)_3 = (021)_3 = (0021)_3 = \cdots \\
8 &= (22)_3 = (022)_3 = (0022)_3 = \cdots \\
9 &= (100)_3 = (0100)_3 = \cdots \\
10 &= (101)_3 = (0101)_3 = \cdots
\end{aligned}$$

It follows that:

$$\begin{array}{lll}
A_0 = \{0\}, & B_0 = \emptyset, & C_0 = \{0\}. \\
A_1 = \{0, 1, 2\}, & B_1 = \{1\}, & C_1 = \{0, 2\}. \\
A_2 = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}, & B_2 = \{1, 3, 4, 5, 7\}, & C_2 = \{0, 2, 6, 8\}.
\end{array}$$

Let r_n denote the size of C_n, s_n the sum of elements of B_n, and t_n the sum of elements of C_n. For example, $r_2 = 4$, $s_2 = 1 + 3 + 4 + 5 + 7 = 20$, and $t_2 = 0 + 2 + 6 + 8 = 16$.

(a) Prove that $r_n = 2^n$ for all $n \geqslant 0$.

Solution Represent each element of A_n as a string of exactly n ternary digits $0, 1, 2$. For $n \geqslant 1$, the set A_n is the disjoint union of $0A_{n-1}$, $1A_{n-1}$ and $2A_{n-1}$ each containing 3^{n-1} elements. The elements of $1A_{n-1}$ contain 1 and hence do not belong to C_n. On the other hand, each of $0A_{n-1}$ and $2A_{n-1}$ contains exactly r_{n-1} integers having no 1 in the ternary representation. It then follows that $r_n = 2r_{n-1}$ for $n \geqslant 1$. Finally, $r_0 = 1$. By repeated substitution, one can easily derive that $r_n = 2^n$ for all $n \geqslant 0$.

(b) Deduce that t_n satisfies the recurrence relation:

$$\begin{aligned}
t_0 &= 0, \\
t_n &= 2t_{n-1} + \frac{1}{3} \times 6^n \quad \text{for all} \quad n \geqslant 1.
\end{aligned}$$

Solution The subset $0A_{n-1}$ contributes t_{n-1} to the sum t_n. The subset $1A_{n-1}$ contributes nothing to the sum t_n. Finally, the subset $2A_{n-1}$ contributes $2 \times 3^{n-1} \times r_{n-1} + t_{n-1}$ to the sum t_n.

(c) Solve the above recurrence relation in order to derive that $t_n = \frac{1}{2}(6^n - 2^n)$ for all $n \geqslant 0$.

Solution The characteristic equation $x - 2 = 0$ has a simple root 2. Therefore, a particular solution is of the form $u6^n$. Plugging in this solution in the recurrence gives $u6^n = 2u6^{n-1} + 2 \times 6^{n-1}$, i.e., $6u = 2u + 2$, i.e., $u = \frac{1}{2}$. A general solution is of the form $t_n = v2^n + \frac{1}{2} \times 6^n$. The initial condition gives $t_0 = 0 = v + \frac{1}{2}$, i.e., $v = -\frac{1}{2}$. Therefore, $t_n = \frac{1}{2}(6^n - 2^n)$.

(d) Conclude that $s_n = \frac{1}{2}(9^n - 6^n - 3^n + 2^n)$ for all $n \geqslant 0$.

Solution We have $s_n + t_n = 0 + 1 + 2 + \cdots + (3^n - 1) = 3^n(3^n - 1)/2 = \frac{1}{2}(9^n - 3^n)$. Plugging in the formula for t_n as derived in the last part yields $s_n = \frac{1}{2}(9^n - 3^n) - \frac{1}{2}(6^n - 2^n) = \frac{1}{2}(9^n - 6^n - 3^n + 2^n)$.

4. Let $k \in \mathbb{N}$, $S = \{1, 2, \ldots, k\}$, and $A = \mathcal{P}(S) \setminus \{\emptyset\}$, where $\mathcal{P}(S)$ denotes the power set of S, and $\emptyset$ denotes the empty set. In other words, the set A comprises all non-empty subsets of $\{1, 2, \ldots, k\}$. For each $a \in A$ denote by $\min(a)$ the smallest element of a (notice that here a is a set).

(a) Define a relation ρ on A as follows: $a \ \rho \ b$ if and only if $\min(a) = \min(b)$. Prove that ρ is an equivalence relation on A.

Solution [Reflexive] For any $a \in A$ we have $\min(a) = \min(a)$.

[Symmetric] For any $a, b \in A$, if $\min(a) = \min(b)$, then $\min(b) = \min(a)$.

[Transitive] For any $a, b, c \in A$, if $\min(a) = \min(b)$ and $\min(b) = \min(c)$, then $\min(a) = \min(c)$.

(b) What is the size of the quotient set A/ρ?

Solution Any two non-empty subsets of S having the same minimum element are related. On the other hand, two subsets of S having different minimum elements are not related. Therefore, each equivalence class of ρ has a one-to-one correspondence with an element of S (the minimum element of every member in the class). Since S contains k elements, there are exactly k equivalence classes, i.e., the size of A/ρ is k.

(c) Define a relation σ on A as follows: $a \ \sigma \ b$ if and only if either $a = b$ or $\min(a) < \min(b)$. Prove that σ is a partial order on A.

Solution [Reflexive] By definition, every element is related to itself.

[Antisymmetric] Take two elements $a, b \in A$. Suppose that $a \ \sigma \ b$ and $b \ \sigma \ a$. If $a \neq b$, then by definition, $\min(a) < \min(b)$ and $\min(b) < \min(a)$, which is impossible. So we must have $a = b$.

[Transitive] Suppose $a \ \sigma \ b$ and $b \ \sigma \ c$ for some $a, b, c \in A$. If $a = b$ or $b = c$, then clearly $a \ \sigma \ c$. So suppose that $a \neq b$ and $b \neq c$. But then $\min(a) < \min(b)$ and $\min(b) < \min(c)$. This implies that $\min(a) < \min(c)$, i.e., $a \ \sigma \ c$.

(d) Is σ also a total order on A?

Solution No! Take $k \geqslant 2$. The sets $\{1\}$ and $\{1, 2\}$ are distinct, but have the same minimum element, and are, therefore, not comparable.

(e) What is the total number of antisymmetric relations on a finite set of size n?

Solution Let X be a set of size n and R an arbitrary antisymmetric relation on X. For each $x \in X$ there are two choices for the diagonal element (x, x): either include it in R or not. Both the choices are compatible with antisymmetry. So take two different elements $x, y \in X$. Antisymmetry demands that one of the following must be true:

(1) Neither (x, y) nor (y, x) belongs to R.
(2) (x, y) belongs to R, but (y, x) does not.
(3) (y, x) belongs to R, but (x, y) does not.

There are $\binom{n}{2} = n(n-1)/2$ ways of choosing two distinct elements x, y from X. Therefore, the total number of antisymmetric relations on X is $2^n \times 3^{n(n-1)/2}$.