CS21001 Discrete Structures, Autumn 2006

Exercise set 2

Solve the following exercises from the text book:

Section 3.1: 22,26,34.
Section 3.2: 12,16.
Section 3.3: 16,22,24.
Section 3.5: 8,20,22,26,34.

Additional exercises

1. Let n be an odd positive integer and $\pi_1, \pi_2, \ldots, \pi_n$ a permutation of $1, 2, \ldots, n$. Prove that the product $\prod_{i=1}^{n} (i - \pi_i)$ is even. (**Hint:** Look at $\pi_1, \pi_3, \pi_5, \ldots, \pi_n$.) Show that the result need not hold if n is even.

2. Let $a_1, a_2, \ldots, a_n$ be positive integers with $\sum_{i=1}^{n} a_i < 2^n - 1$. Prove that there exist distinct disjoint non-empty subsets A, B of $\{a_1, a_2, \ldots, a_n\}$ with the property that $\sum_{a \in A} a = \sum_{b \in B} b$.

3. Prove the identity: $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}$.

4. Count the solutions of the following:

(a) $x_1 + x_2 + x_3 + x_4 = 56$ with non-negative integers x_1, x_2, x_3, x_4.
(b) $x_1 + x_2 + x_3 + x_4 = 56$ with positive integers x_1, x_2, x_3, x_4.
(c) $x_1 + x_2 + x_3 + x_4 = 56$ with integers $x_1 \geqslant 1$, $x_2 \geqslant 2$, $x_3 \geqslant 3$, $x_4 \geqslant 4$.
(d) $x_1 + x_2 + x_3 + x_4 \leqslant 56$ with non-negative integers x_1, x_2, x_3, x_4. (**Hint:** Introduce x_5.)
(e) $x_1 + x_2 + x_3 + x_4 \leqslant 56$ with integers $x_1 \geqslant 1$, $x_2 \geqslant 2$, $x_3 \geqslant 3$, $x_4 \geqslant 4$.
(f) $x_1 + x_2 + x_3 + x_4 \geqslant 56$ with integers $x_1 \leqslant 11$, $x_2 \leqslant 22$, $x_3 \leqslant 33$, $x_4 \leqslant 44$. (**Hint:** Take $y_i := 11i - x_i$.)

5. The principle of inclusion and exclusion is often stated in the following form. Prove it.

Let X be a set, $\mathcal{P}(X)$ the power set of X, and let $f, g : \mathcal{P}(X) \to \mathbb{R}$ be functions such that $f(A) = \sum_{S \subseteq A} g(S)$ for all subsets A of X. Then $g(A) = \sum_{S \subseteq A} (-1)^{|A|-|S|} f(S)$ for all subsets A of X.

6. Solve the following recurrence relations:

(a) $a_0 = 2$, $a_n = 5a_{n-1} + (n^2 + n + 1)$ for $n \geqslant 1$.
(b) $a_0 = 2$, $a_n = 5a_{n-1} + (n^2 + n + 1)2^n$ for $n \geqslant 1$.
(c) $a_0 = 2$, $a_1 = 3$, $a_n = 2(a_{n-1} + a_{n-2} + 2^n)$ for $n \geqslant 2$.
(d) $a_0 = 2$, $a_1 = 3$, $a_n = 4(a_{n-1} - a_{n-2} + 2^n)$ for $n \geqslant 2$.
(e) $a_0 = 2$, $a_1 = 3$, $a_n = 4(a_{n-1} - a_{n-2} + n2^{2n-1})$ for $n \geqslant 2$.

7. Reduce the following recurrence relations to standard forms and solve:

(a) $a_0 = 2$, $a_1 = 3$, $a_n = a_{n+2} - a_{n+1} - n$ for all $n \geqslant 0$.
(b) $a_0 = 2$, $a_1 = 3$, $4a_{n+1} + 8a_n - 5a_{n-1} = 2^n$ for all $n \geqslant 1$.
(c) $a_0 = 2$, $a_1 = 3$, $a_n = a_{n-1} + 12(a_{n-2} + 2^{n-2})$ for all $n \geqslant 2$.
(d) $a_0 = 2$, $a_n^3 = a_{n-1}(3a_n^2 - 3a_n a_{n-1} + a_{n-1}^2) + n^3$ for all $n \geqslant 1$.

(e) $a_0 = 2$, $a_1 = 3$, $2a_n a_{n-2} - 2a_{n-1}^2 - 3a_{n-1}a_{n-2} = 0$ for all $n \geqslant 2$.
(f) $a_0 = 2$, $a_1 = 3$, $2^{a_n} = 4^n \times 16^{a_{n-2}}$ for all $n \geqslant 2$.

8. Find big-O estimates for the following positive-integer-valued increasing functions $f(n)$.

(a) $f(n) = 125f(n/4) + 2n^3$ whenever $n = 4^k$ for $k \in \mathbb{Z}^+$.
(b) $f(n) = 125f(n/5) + 2n^3$ whenever $n = 5^k$ for $k \in \mathbb{Z}^+$.
(c) $f(n) = 125f(n/6) + 2n^3$ whenever $n = 6^k$ for $k \in \mathbb{Z}^+$.

9. Let $f(n)$ be an increasing positive-real-valued function of a non-negative integer variable n. Give a big-O estimate of $f(n)$ for each of the following cases:

(a) $f(n) = 2f(\sqrt{n}) + 1$ whenever n is a perfect square bigger than 1.
(b) $f(n) = 2f(\sqrt{n}) + \log n$ whenever n is a perfect square bigger than 1.
(c) $f(n) = 2f(\sqrt{n}) + \log^2 n$ whenever n is a perfect square bigger than 1.
(d) $f(n) = af(\sqrt[b]{n}) + c(\log n)^d$ whenever n is a perfect b-th power bigger than 1. Here $a, b \in \mathbb{N}$, $a \geqslant 1$, $b \geqslant 2$, $c, d \in \mathbb{R}$, $c > 0$ and $d \geqslant 0$.