CS21001 Discrete Structures, Autumn 2006

End-semester examination

Total marks: 100 | November 20, 2006 (AN): S-302 (B) | **Duration:** 3 hours

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[*Answer* <u>*all*</u> *questions*]

1 Consider the following C function:

```
unsigned int f ( unsigned int n )
{
   if (n == 0) return 0;
   if (n % 2 == 1) return 0;
   return 1 + f(n*(n+1)/2);
}
```

Denote by $f(n)$ the value returned by the above function upon input n.

(a) Derive the values of $f(2), f(3), f(4), f(5)$, and $f(6)$. **(5)**

(b) Notice that if n is large, the computation of $n(n+1)/2$ (argument for the recursive call) may lead to an overflow. Assume that for some value of n, no overflow occurs during any of the recursive calls. Argue that in this case the function terminates after $\mathrm{O}(\log n)$ number of recursive calls, where n is the argument of f in the outermost call. **(5)**

(c) Suppose that your machine supports 32-bit unsigned integers and during a multiplication ab of unsigned integers a, b, the least significant 32 bits of the actual product is returned. This means that if there is no overflow during the multiplication, you get the correct product. In case of an overflow, you obtain the least significant 32 bits of the correct product. Justify that in this case too, the function makes $\mathrm{O}(\log n)$ number of recursive calls, where n is the argument of f in the outermost call. **(5)**

(d) Describe (with proper justification) what $f(n)$ returns for a positive integer n. **(5)**

(e) Deduce that the running time of the above function is $\mathrm{O}(\log n)$. **(5)**

2 Argue (with proper justification) which of the following sets is/are countable. **(5×5)**

(a) The set $S_1 = \{n \in \mathbb{R} \mid 3^n + 2^n = 35\}$. (Here $\mathbb{R}$ is the set of real numbers.)

(b) The set $S_2 = \{2, 3, 5, 7, \ldots\}$ of all (positive) prime integers.

(c) The set S_3 of all (finite) subsets of $\mathbb{N}$ whose sizes are odd.

(d) The set S_4 of all subsets of $\mathbb{N}$ containing no odd integers.

(e) The set S_5 of all functions $f : \mathbb{N} \to \mathbb{Z}$ with the property that $f(n) = 0$ except for finitely many $n \in \mathbb{N}$.

3 In this exercise we work in the semigroup $\mathbb{N}$ under integer multiplication. Define a relation ρ on $\mathbb{N}$ as $a \; \rho \; b$ if and only if a has the same set of prime divisors as b. For example, 5 is related to $25 = 5^2$, $12 = 2^2 \times 3$ is related to $54 = 2 \times 3^3$, but 12 is not related to $16 = 2^4$ nor to $180 = 2^2 \times 3^2 \times 5$.

(a) Prove that ρ is a congruence relation on $\mathbb{N}$. **(5)**

(b) Find the equivalence classes of $1, 2, 3, 4$ and 6. **(5)**

(c) A non-zero integer is called *square-free* if it is not divisible by the square of a prime number. Prove that each equivalence class in $\mathbb{N}/\rho$ contains a unique square-free integer, and that these unique square-free integers are different in distinct equivalence classes. **(5)**

4 **(a)** Let G be the set of all invertible (i.e., non-singular) 2×2 matrices with real entries. Prove that G is a group under matrix multiplication. **(5)**

(b) Define the *center* $Z(G)$ of G as:

$$Z(G) = \{A \in G \mid AP = PA \text{ for all } P \in G\}.$$

Prove that $Z(G)$ is a normal subgroup of G. **(5)**

(c) Derive that $Z(G) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{R}, a \neq 0 \right\}$. **(5)**

(d) A matrix $A \in G$ is said to be *similar* to a matrix $B \in G$ if $B = PAP^{-1}$ for some $P \in G$. Prove that similarity is an equivalence relation on G. **(5)**

(e) Prove or disprove: Similarity is a congruence relation on G. **(5)**

(f) For any fixed $P \in G$, define the map $f_P : G \to G$ as $f_P(A) = PAP^{-1}$. Prove that f_P is a group isomorphism. **(5)**

(g) Prove that f_P is the identity map on G if and only if $P \in Z(G)$. **(5)**