CS21001 Discrete Structures, Autumn 2006

Class test 2

Total marks: 30 November 15, 2006 (6:00-7:00pm) **Duration:** 1 hour

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Roll No: ____________________ **Name:** ____

1 **(a)** In this part, we show that $\mathbb{N} \times \mathbb{N}$ is equinumerous with $\mathbb{N}$. To that effect, define the map $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ as follows. Any $n \in \mathbb{N}$ can be written uniquely as $n = 2^s t$, where s is a non-negative integer and t is a positive odd integer. For this n, define $f(n) = (s + 1, (t + 1)/2)$. Prove that f is a bijection. **(5)**

(b) In this part, we show that $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is equinumerous with $\mathbb{N}$. As in the previous part, one can construct an explicit bijection between $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$. It is, however, easier to use the Cantor-Schröder-Bernstein theorem.

(i) Propose an explicit injective map $g : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. **(5)**

$g(a, b, c) =$ ____

(ii) Propose an explicit injective map $h : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. **(5)**

$h(n) =$ ____

2 Let $(S, *)$ be a semigroup and $a \in S$. Recall that the sub-semigroup generated by a is the set $\langle a \rangle = \{a * a * \cdots * a \ (n \text{ times}) \mid n > 0\}$. S is called *cyclic* if $S = \langle a \rangle$ for some $a \in S$. Justify which of the following semigroups is/are cyclic.

(a) $\mathbb{N}$ under integer multiplication. **(5)**

(b) $\mathbb{Z}$ under integer addition. **(5)**

(c) $\mathbb{Z}_n$ under addition modulo n (for some arbitrary $n \in \mathbb{N}$). **(5)**