CS21001 Discrete Structures, Autumn 2006

Class test 2: Solutions

Total marks: 30 November 15, 2006 (6:00-7:00pm) **Duration:** 1 hour

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Roll No: ____________ **Name:** ____________________________

1 **(a)** In this part, we show that $\mathbb{N} \times \mathbb{N}$ is equinumerous with $\mathbb{N}$. To that effect, define the map $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ as follows. Any $n \in \mathbb{N}$ can be written uniquely as $n = 2^s t$, where s is a non-negative integer and t is a positive odd integer. For this n, define $f(n) = (s+1, (t+1)/2)$. Prove that f is a bijection. **(5)**

Solution Suppose that $f(n) = (s+1, (t+1)/2)$ and $f(n') = (s'+1, (t'+1)/2)$ are equal, i.e., $s+1 = s'+1$ and $(t+1)/2 = (t'+1)/2$, i.e., $s = s'$ and $t = t'$. But then $n = 2^s t = 2^{s'} t' = n'$. Thus f is injective.

Now take any $(a, b) \in \mathbb{N} \times \mathbb{N}$. Let $s = a - 1$, $t = 2b - 1$, and $n = 2^s t$. Then s is a non-negative integer and t a positive odd integer, and so $n \in \mathbb{N}$. But then $f(n) = (s+1, (t+1)/2) = (a, b)$. That is, f is surjective.

(b) In this part, we show that $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is equinumerous with $\mathbb{N}$. As in the previous part, one can construct an explicit bijection between $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$. It is, however, easier to use the Cantor-Schröder-Bernstein theorem.

(i) Propose an explicit injective map $g : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. **(5)**

$g(a, b, c) =$ $2^a \times 3^b \times 5^c$.

(Injectivity of g follows from the unique factorization property of integers.)

(ii) Propose an explicit injective map $h : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. **(5)**

$h(n) =$ $(n, 1, 1)$.

2 Let $(S, *)$ be a semigroup and $a \in S$. Recall that the sub-semigroup generated by a is the set $\langle a \rangle = \{a * a * \cdots * a \ (n \text{ times}) \mid n > 0\}$. S is called *cyclic* if $S = \langle a \rangle$ for some $a \in S$. Justify which of the following semigroups is/are cyclic.

(a) $\mathbb{N}$ under integer multiplication. **(5)**

Solution [No] Every positive integer cannot be written as a power of a fixed (positive) integer. More precisely, suppose $\mathbb{N} = \langle a \rangle$ for some $a \in \mathbb{N}$. But then $2 = a^i$ and $3 = a^j$ for some $i, j > 0$. But then a divides both 2 and 3, whereas $2, 3$ are coprime. Thus $a = 1$, and consequently $\langle a \rangle = \{1\} \neq \mathbb{N}$, a contradiction.

(b) $\mathbb{Z}$ under integer addition. **(5)**

Solution [No] Consider $\langle a \rangle$ for some $a \in \mathbb{Z}$. We have $\langle 0 \rangle = \{0\}$. So assume that $a \neq 0$. If $a > 0$, then $\langle a \rangle$ contains only positive integers. On the other hand, if $a < 0$, then $\langle a \rangle$ contains only negative integers. In all these cases, $\langle a \rangle$ is a proper subset of $\mathbb{Z}$.

(c) $\mathbb{Z}_n$ under addition modulo n (for some arbitrary $n \in \mathbb{N}$). **(5)**

Solution [Yes] $\mathbb{Z}_n$ is generated by (the equivalence class of) 1. Notice that

$$\begin{array}{rcl}
1 & \equiv & 1 \pmod{n}, \\
2 & \equiv & 1 + 1 \pmod{n}, \\
3 & \equiv & 1 + 1 + 1 \pmod{n}, \\
 & \cdots & \\
n - 1 & \equiv & 1 + 1 + \cdots + 1 \ (n - 1 \text{ times}) \pmod{n}, \text{ and} \\
0 & \equiv & 1 + 1 + \cdots + 1 \ (n \text{ times}) \pmod{n}.
\end{array}$$