CS21001 Discrete Structures, Autumn 2006

Class test 1

Total marks: 30 | September 12, 2006 (6:00-7:00pm) | **Duration:** 1 hour

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1. Let $P(x), Q(x)$ be predicates involving an integer-valued variable x. Prove or disprove: $\forall x\,[P(x) \Rightarrow Q(x)]$ is logically equivalent to $\forall x\,[P(x)] \Rightarrow \forall x\,[Q(x)]$. **(5)**

Solution False: Let $P(x)$ be the predicate "x is even", and $Q(x)$ the predicate "$x+1$ is even". Then, for example, $P(2) \Rightarrow Q(2)$ is false, and so $\forall x\,[P(x) \Rightarrow Q(x)]$ is false. On the other hand, $\forall x\,[P(x)]$ is false, since all integers are not even, and so $\forall x\,[P(x)] \Rightarrow \forall x\,[Q(x)]$ is true.

2. The following recursive function takes as argument an array A of integers and its size $n \geqslant 1$.

```
int f ( int A[], unsigned int n )
{
   if (n == 1) return 0;
   if (n == 2) return A[1];
   return f(A,2) + f(&A[1],n-1) + f(&A[2],n-2);
}
```

(a) Let the element at index i in the array A be denoted by a_i. Prove by induction on n that the function returns $F_0a_0 + F_1a_1 + F_2a_2 + \cdots + F_{n-1}a_{n-1}$ for all $n \geqslant 1$, where F_i is the i-th Fibonacci number. **(10)**

Solution [Basis] For $n = 1$ the function returns $0 = F_0a_0$ (since $F_0 = 0$). For $n = 2$ the function returns $a_1 = F_0a_0 + F_1a_1$ (since $F_0 = 0$ and $F_1 = 1$).

[Induction] Take $n \geqslant 3$, and assume that the function returns the value mentioned above for all arrays of size $< n$. Now consider the situation that an array of size n is passed to the function. By induction, the first recursive call returns

$$F_0a_0 + F_1a_1,$$

the second recursive call returns

$$\begin{aligned} & F_0a_1 + F_1a_2 + F_2a_3 + \cdots + F_{n-2}a_{n-1} \\ = \quad & F_1a_2 + F_2a_3 + \cdots + F_{n-2}a_{n-1}, \qquad \text{[notice that } F_0 = 0] \end{aligned}$$

and the third recursive call returns

$$F_0a_2 + F_1a_3 + F_2a_4 + \cdots + F_{n-3}a_{n-1}.$$

The sum of these returned values is

$$\begin{aligned} & F_0a_0 + F_1a_1 + (F_1 + F_0)a_2 + (F_2 + F_1)a_3 + (F_3 + F_2)a_4 + \cdots + (F_{n-2} + F_{n-3})a_{n-1} \\ = \quad & F_0a_0 + F_1a_1 + F_2a_2 + F_3a_3 + F_4a_4 + \cdots + F_{n-1}a_{n-1}. \end{aligned}$$

(b) Let T_n denote the running time of the above function on an array of size n. Write a recurrence relation for T_n. Also supply the requisite number of initial conditions. **(5)**

Solution

$$\begin{aligned} T_1 &= a, \\ T_2 &= b, \\ T_n &= T_2 + T_{n-1} + T_{n-2} + c \\ &= T_{n-1} + T_{n-2} + d \quad \text{for } n \geqslant 3\,. \end{aligned}$$

Here a, b, c, d are positive constants. Notice that T_2 is a constant, so $d = c + T_2$ is constant too.

(c) Solve the above recurrence relation to obtain an explicit formula for T_n. Conclude that $T_n = \Theta(\phi^n)$, where ϕ is the golden ratio. **(10)**

Solution The characteristic equation $x^2 = x + 1$ has roots $x = \frac{1 \pm \sqrt{5}}{2}$. So a particular solution is of the form $w \times 1^n$. Substitution gives $w = w + w + d$, i.e., $w = -d$. Therefore, a general solution is of the form

$$T_n = \left(\frac{1+\sqrt{5}}{2}\right)^n u + \left(\frac{1-\sqrt{5}}{2}\right)^n v - d,$$

where u, v are constants to be determined from the initial conditions:

$$\begin{aligned} T_1 = a &= \left(\frac{1+\sqrt{5}}{2}\right) u + \left(\frac{1-\sqrt{5}}{2}\right) v - d, \\ T_2 = b &= \left(\frac{3+\sqrt{5}}{2}\right) u + \left(\frac{3-\sqrt{5}}{2}\right) v - d. \end{aligned}$$

Solving this system gives:

$$\begin{aligned} u &= \frac{1}{2\sqrt{5}}\Big[(3-\sqrt{5})a + (-1+\sqrt{5})b + 2d\Big], \\ v &= \frac{1}{2\sqrt{5}}\Big[(-3-\sqrt{5})a + (1+\sqrt{5})b - 2d\Big]. \end{aligned}$$

Notice that the exact values of the constants u, v, w are not very important here. It suffices to know the form of the solution. In particular, since a, b, d are positive, it follows that $u > 0$. Also $|\frac{1-\sqrt{5}}{2}| < 1$. Consequently, $T_n = \Theta\left(\left(\frac{1+\sqrt{5}}{2}\right)^n\right)$.