CS21001 Discrete Structures, Autumn 2005

Mid-semester examination

Total marks: 60 September 16, 2005 **Duration:** 2 hours

[*Answer any four questions*]

1. Let T_n be a sequence of positive integers defined recursively as:

$$\begin{aligned} T_0 &= 2, \\ T_n &= T_{n-1}^2 + T_{n-2}^2 + \cdots + T_1^2 + T_0^2 \quad \text{for all } n \geqslant 1. \end{aligned}$$

Prove the following assertions. You may use induction on n, whenever necessary.

(a) $T_n = T_{n-1}(T_{n-1} + 1)$ for all $n \geqslant 2$. **(5)**

(b) $T_n \geqslant 2^{2^n}$ for all $n \in \mathbb{N}_0$. **(5)**

(c) $T_n \leqslant 2^{3^n}$ for all $n \in \mathbb{N}_0$. **(5)**

2. A partial order ρ on a set A is called a *total order* (or a *linear order*) if for any two different $a, b \in A$ either $a \;\rho\; b$ or $b \;\rho\; a$. Justify which of the following relations ρ, σ, τ on $\mathbb{N}$ are total orders. (For each of the relations ρ, σ, τ, first determine whether the relation is a partial order, and if so, whether it is a total order.)

(a) $a \;\rho\; b$ if and only if $a \leqslant b + 1701$. **(5)**

(b) $a \;\sigma\; b$ if and only if $a \geqslant b + 1701$. **(5)**

(c) $a \;\tau\; b$ if and only if either $u < v$ or $u = v$ and $x \leqslant y$, where $a = 2^u x$ and $b = 2^v y$ with x and y odd. **(5)**

3. **(a)** Prove that the set of all subsets of $\mathbb{N}$ is uncountable. **(5)**

(b) Prove that the set of all *finite* subsets of $\mathbb{N}$ is countable. **(5)**

(c) Prove that the set of all *infinite* subsets of $\mathbb{N}$ is uncountable. **(5)**

4. There are 8 red balls, 12 blue balls, 16 green balls and 20 yellow balls in a bag.

(a) What is the minimum number of balls you must take out from the bag, so that you are guaranteed to get 6 balls of the same color? **(5)**

(b) What is the minimum number of balls you must take out from the bag, so that you are guaranteed to get 10 balls of the same color? **(5)**

(c) What is the minimum number of balls you must take out from the bag, so that you are guaranteed to get at least one ball of each color? **(5)**

5. Let $A_n = \{1, 2, 3, \ldots, n\}$. Recall that the total number of partitions of A_n is the n-th Bell number B_n. Let B'_n denote the total number of partitions of A_n for which any pair of consecutive integers (i and $i + 1$) does not belong to the same subset of a partition. Prove that $B'_n = B_{n-1}$ for all $n \geqslant 1$. **(15)**

6. Solve the following recurrence relation to find an explicit formula for the sequence T_n: **(15)**

$$\begin{aligned} T_0 &= 1, \\ T_1 &= 2, \\ T_2 &= 28, \\ T_n &= 5T_{n-1} - 3T_{n-2} - 9T_{n-3} \quad \text{for all } n \geqslant 3. \end{aligned}$$

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