CS21001 Discrete Structures, Autumn 2005

Mid-semester examination : Solution

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1. *Let T_n be a sequence of positive integers defined recursively as:*

$$\begin{aligned} T_0 &= 2, \\ T_n &= T_{n-1}^2 + T_{n-2}^2 + \cdots + T_1^2 + T_0^2 \quad \text{for all } n \geqslant 1. \end{aligned}$$

Prove the following assertions. You may use induction on n, whenever necessary.

(a) $T_n = T_{n-1}(T_{n-1}+1)$ *for all* $n \geqslant 2$.

Solution For $n \geqslant 2$ we have:

$$\begin{aligned} T_n &= T_{n-1}^2 + T_{n-2}^2 + T_{n-3}^2 + \cdots + T_1^2 + T_0^2, \\ T_{n-1} &= \phantom{T_{n-1}^2 + {}} T_{n-2}^2 + T_{n-3}^2 + \cdots + T_1^2 + T_0^2, \end{aligned}$$

so that $T_n - T_{n-1} = T_{n-1}^2$, i.e., $T_n = T_{n-1}(T_{n-1}+1)$.

(b) $T_n \geqslant 2^{2^n}$ *for all* $n \in \mathbb{N}_0$.

Solution [Induction on n] $T_0 = 2 = 2^{2^0}$. So assume $n \geqslant 1$ and that $T_{n-1} \geqslant 2^{2^{n-1}}$. The recurrence implies $T_n \geqslant T_{n-1}^2 \geqslant \left(2^{2^{n-1}}\right)^2 = 2^{2^n}$.

(c) $T_n \leqslant 2^{3^n}$ *for all* $n \in \mathbb{N}_0$.

Solution [Induction on n] $T_0 = 2 = 2^{3^0}$ and $T_1 = T_0^2 = 4 \leqslant 2^{3^1}$. So take $n \geqslant 2$ and assume that $T_{n-1} \leqslant 2^{3^{n-1}}$. But then by Part (a) we have $T_n = T_{n-1}(T_{n-1}+1) \leqslant T_{n-1}(T_{n-1}^2)$ (since $x+1 \leqslant x^2$ for all $x \geqslant 2$), i.e., $T_n \leqslant T_{n-1}^3 \leqslant \left(2^{3^{n-1}}\right)^3 = 2^{3^n}$.

2. *A partial order ρ on a set A is called a total order (or a linear order) if for any two different $a, b \in A$ either $a \;\rho\; b$ or $b \;\rho\; a$. Justify which of the following relations ρ, σ, τ on $\mathbb{N}$ are total orders. (For each of the relations ρ, σ, τ, first determine whether the relation is a partial order, and if so, whether it is a total order.)*

(a) $a \;\rho\; b$ *if and only if* $a \leqslant b + 1701$.

Solution $1 \;\rho\; 2$ and $2 \;\rho\; 1$, but $1 \neq 2$, i.e., ρ is not antisymmetric and so not a partial order too.

(b) $a \;\sigma\; b$ *if and only if* $a \geqslant b + 1701$.

Solution Let $a \;\sigma\; b$ and $b \;\sigma\; a$. This implies that $a \geqslant b + 1701 \geqslant (a + 1701) + 1701 = a + 3402$. This is absurd. So we cannot have both $a \;\sigma\; b$ and $b \;\sigma\; a$ simultaneously, i.e., σ is antisymmetric. If $a \;\sigma\; b$ and $b \;\sigma\; c$, we have $a \geqslant b + 1701 \geqslant (c + 1701) + 1701 \geqslant c + 1701$, i.e., σ is transitive too. Therefore, σ is a partial order on $\mathbb{N}$. But σ is not a total order on $\mathbb{N}$, since neither $(1, 1701)$ nor $(1701, 1)$ belongs to σ.

(c) $a \;\tau\; b$ *if and only if either* $u < v$ *or* $u = v$ *and* $x \leqslant y$, *where* $a = 2^u x$ *and* $b = 2^v y$ *with* x *and* y *odd.*

Solution Let $a = 2^u x$ and $b = 2^v y$ (with x, y odd) satisfy $a \;\tau\; b$ and $b \;\tau\; a$. We cannot have $u < v$ and $v \leqslant u$ simultaneously. So $u = v$. But then $x \leqslant y$ and $y \leqslant x$, implying $x = y$, i.e., $a = b$. So τ is anti-symmetric. Now suppose $a \;\tau\; b$ and $b \;\tau\; c$, where $a = 2^u x$, $b = 2^v y$ and $c = 2^w z$ with x, y, z odd. We have $u \leqslant v$ and $v \leqslant w$, i.e., $u \leqslant w$. If $u < w$, then $a \;\tau\; c$. On the other hand, $u = w$ implies $u = v = w$. But then $x \leqslant y$ and $y \leqslant z$, so that $x \leqslant z$, i.e., $a \;\tau\; c$. So τ is a partial order. Finally, let $a = 2^u x$ and $b = 2^v y$ be two different integers. We then have either $u \neq v$ or $x \neq y$ (or both). If $u < v$, then $a \;\tau\; b$. If $u > v$, then $b \;\tau\; a$. If $u = v$, then $a \;\tau\; b$ or $b \;\tau\; a$ according as whether $x < y$ or $x > y$. Thus, τ is a total order.

3. (a) *Prove that the set of all subsets of $\mathbb{N}$ is uncountable.*

Solution No set is in bijective correspondence with its power set.

(b) *Prove that the set of all finite subsets of* $\mathbb{N}$ *is countable.*

Solution Let A denote the set of all finite subsets of $\mathbb{N}$. We write A as the disjoint union $A = \bigcup_{n \in \mathbb{N}_0} A_n$, where A_n comprises subsets of $\mathbb{N}$ of size n. $|A_0| = 1$. For $n \geqslant 1$ the set A_n can be identified with an (infinite) subset of $\mathbb{N}^n$ and so is countable. Since A is the union of countably many finite or countable sets, it is countable.

(c) *Prove that the set of all infinite subsets of* $\mathbb{N}$ *is uncountable.*

Solution Let B denote the set of all infinite subsets of $\mathbb{N}$. $\mathcal{P}(\mathbb{N})$ is the (disjoint) union of A (defined in Part (b)) and B. If B is countable, then $\mathcal{P}(\mathbb{N})$ would be countable too. So B is uncountable.

4. *There are 8 red balls, 12 blue balls, 16 green balls and 20 yellow balls in a bag.*

(a) *What is the minimum number of balls you must take out from the bag, so that you are guaranteed to get 6 balls of the same color?*

Solution The color can be any one of red, blue, green and yellow. By the pigeon-hole principle, choosing k balls from the bag ensures one color to be repeated at least $\lceil k/4 \rceil$ times. The smallest k for which $\lceil k/4 \rceil \geqslant 6$ is 21. On the other hand, drawing 20 balls does not provide the desired guarantee, since we may have 5 balls of each color.

(b) *What is the minimum number of balls you must take out from the bag, so that you are guaranteed to get 10 balls of the same color?*

Solution We cannot draw 10 red balls, since there are only 8 of them. Since 28 the smallest value of k for which $\lceil k/3 \rceil \geqslant 10$, we must draw 28 balls of colors blue, green and yellow in order to have a color repeated at least 10 times. But the drawing may give a number between 0 to 8 of the red balls. These are useless for our goal, but we must remain prepared for drawing them. Thus the required minimum number of balls in this case is $28 + 8 = 36$.

(c) *What is the minimum number of balls you must take out from the bag, so that you are guaranteed to get at least one ball of each color?*

Solution The desired number is $1 + 12 + 16 + 20 = 49$. Drawing 48 balls may give a collection of only the blue, green and yellow balls.

5. *Let* $A_n = \{1, 2, 3, \ldots, n\}$. *Recall that the total number of partitions of* A_n *is the* n*-th Bell number* B_n. *Let* B'_n *denote the total number of partitions of* A_n *for which any pair of consecutive integers (*i *and* $i+1$*) does not belong to the same subset of a partition. Prove that* $B'_n = B_{n-1}$ *for all* $n \geqslant 1$.

Solution Call a partition P of A_n special, if no subset in P contains two consecutive integers. Denote by $S'(n, r)$ the number of special partitions of A_n containing exactly r non-empty subsets. First I prove by induction on n that $S'(n, r) = S(n-1, r-1)$ for all $n \geqslant 1$ and for all r in the range $1 \leqslant r \leqslant n$. For $n = 1$, we have $S'(n, 1) = 1 = S(0, 0)$ and the induction basis is proved. So take $n \geqslant 2$ and $1 \leqslant r \leqslant n$, and assume that the claim holds for $S'(n-1, r')$ for all legitimate indices r'. Consider a special partition of A_{n-1} into $r-1$ non-empty subsets. Adding $\{n\}$ as a singleton gives a special partition of A_n. Then consider a special partition of A_{n-1} into r non-empty subsets. Adding n to any one of the $r-1$ subsets not containing $n-1$ yields a special partition of A_n. All special partitions of A_n into r non-empty subsets can be generated in this way. Consequently,

$$\begin{aligned}
S'(n, r) &= S'(n-1, r-1) + (r-1)S'(n-1, r) \\
&= S(n-2, r-2) + (r-1)S(n-2, r-1) \quad \text{[by the induction hypothesis]} \\
&= S(n-1, r-1) \quad \text{[by the recurrence for Stirling numbers]},
\end{aligned}$$

and the inductive step is established. For $n \geqslant 1$ it then follows that

$$\begin{aligned}
B'_n &= S'(n, 0) + \sum_{r=1}^{n} S'(n, r) = \sum_{r=1}^{n} S'(n, r) \quad \text{[since } S'(n, 0) = 0\text{]} \\
&= \sum_{r=1}^{n} S(n-1, r-1) = \sum_{r-1=0}^{n-1} S(n-1, r-1) = B_{n-1}.
\end{aligned}$$

6. *Solve the following recurrence relation to find an explicit formula for the sequence T_n:*

$$\begin{aligned} T_0 &= 1, \\ T_1 &= 2, \\ T_2 &= 28, \\ T_n &= 5T_{n-1} - 3T_{n-2} - 9T_{n-3} \quad \textit{for all } n \geqslant 3. \end{aligned}$$

Solution The characteristic equations is

$$\begin{aligned} \chi(X) &= X^3 - 5X^2 + 3X + 9 \;=\; X^3 - 3X^2 - 2X^2 + 6X - 3X + 9 \\ &= (X-3)(X^2 - 2X - 3) \;=\; (X-3)(X+1)(X-3) \;=\; (X+1)(X-3)^2 \;=\; 0. \end{aligned}$$

Therefore, a general solution of the given recurrence is:

$$T_n = c(-1)^n + c'3^n + c''n3^n.$$

Plugging in the initial values gives:

$$\begin{aligned} c + c' &= 1, \\ -c + 3c' + 3c'' &= 2, \\ c + 9c' + 18c'' &= 28. \end{aligned}$$

The solution of this system is $c = 25/16$, $c' = -9/16$ and $c'' = 7/4$. Thus we have:

$$T_n = \frac{25}{16}(-1)^n + \left(\frac{7}{4}n - \frac{9}{16}\right)3^n \quad \text{for all } n \in \mathbb{N}_0.$$