CS21001 Discrete Structures, Autumn 2005

Exercise set 3

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1. Let $n, m, r \in \mathbb{N}$ and $x, y \in \mathbb{R}$. Prove that:

(a) $\lfloor nm/r \rfloor \geqslant n \lfloor m/r \rfloor$ and $\lceil nm/r \rceil \leqslant n \lceil m/r \rceil$.

(b) $\lfloor \frac{n}{2} \rfloor - \lfloor -\frac{n}{2} \rfloor = \lceil \frac{n}{2} \rceil - \lceil -\frac{n}{2} \rceil = n$.

(c) $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leqslant \lfloor 2x \rfloor + \lfloor 2y \rfloor$.

2. Let n be a positive integer and p a prime.

(a) Prove that the largest exponent e such that p^e that divides $n!$ is $\sum_{r \in \mathbb{N}} \left\lfloor \frac{n}{p^r} \right\rfloor$.

(b) Write an efficient algorithm that, given n, returns the exact number of trailing 0's in the decimal representation of $n!$.

(c) Prove that the largest exponent e for which p^e divides $\binom{2n}{n}$ is $\sum_{r \in \mathbb{N}} \left(\left\lfloor \frac{2n}{p^r} \right\rfloor - 2 \left\lfloor \frac{n}{p^r} \right\rfloor \right)$.

3. Count the bit-strings of length ten that:

(a) start with 01 and end with 10.
(b) start with 01 and do not end with 10.
(c) neither start with 01 nor end with 10.
(d) contain neither 01 nor 10 as a substring.
(e) contain 01 but not 10 as a substring.
(f) contain both 01 and 10 as substrings.
(g) contain equal number of 0's and 1's.
(h) contain more 0's than 1's.

4. Count the positive integers less than or equal to 1000 that are:

(a) divisible by 5 or 7 (or both).
(b) divisible by both 5 and 7.
(c) divisible by neither 5 nor 7.
(d) divisible by 5 but not by 7.
(e) divisible by 6 or 8 (or both).
(f) divisible by both 6 and 8.
(g) divisible by neither 6 nor 8.
(h) divisible by 6 but not by 8.

5. Prove that if five points are placed inside an equilateral triangle of side 1 cm, there exist two of these points, that are no more than $1/2$ cm apart.

6. Let n be an odd positive integer and π a permutation of $1, 2, \ldots, n$, i.e., a bijective function $A \to A$, where $A := \{1, 2, \ldots, n\}$. Prove that the product $\prod_{i=1}^{n} (i - \pi(i))$ is even. (**Hint:** Look at the $(n+1)/2$ images $\pi(1), \pi(3), \pi(5), \ldots, \pi(n)$.) Show that the result need not hold if n is even.

7. Let $A \subseteq \{1, 2, \ldots, 2n\}$ with $|A| = n + 1$. Prove that:

(a) There exist $x_1, y_1 \in A$ such that $x_1 - y_1 = 1$.
(b) There exist $x_2, y_2 \in A$ such that $x_2 - y_2 = n$.
* **(c)** There exist $x_3, y_3 \in A$ such that $\gcd(x_3, y_3) = 1$.

8. Let $a_1, a_2, \ldots, a_n$ be positive integers with $\sum_{i=1}^{n} a_i < 2^n - 1$. Prove that there exist distinct disjoint non-empty subsets A, B of $\{a_1, a_2, \ldots, a_n\}$ with the property that $\sum_{a \in A} a = \sum_{b \in B} b$.

* **9.** Let $f(X)$ be a polynomial with integer coefficients such that $f(a) = f(b) = f(c) = 2$ for three distinct integers a, b, c. Prove that $f(n) \neq 1$ for all $n \in \mathbb{Z}$. (**Hint:** First show that $m - n$ divides $f(m) - f(n)$ for any two integers m, n.)

10. Prove the following identities involving binomial coefficients:

(a) $$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}.$$

(b) $\binom{m}{r} = \sum_{k=0}^{2r} (-1)^{r+k} \binom{m}{k} \binom{m}{2r-k}$. **(c)** $\binom{m}{r} = \sum_{k=0}^{\lfloor r/2 \rfloor} (-1)^k \binom{m}{k} \binom{m+r-2k-1}{r-2k}$.

11. Count the solutions of the following:

(a) $x_1 + x_2 + x_3 + x_4 = 56$ with non-negative integers x_1, x_2, x_3, x_4.
(b) $x_1 + x_2 + x_3 + x_4 = 56$ with positive integers x_1, x_2, x_3, x_4.
(c) $x_1 + x_2 + x_3 + x_4 = 56$ with integers $x_1 \geqslant 1$, $x_2 \geqslant 2$, $x_3 \geqslant 3$, $x_4 \geqslant 4$.
(d) $x_1 + x_2 + x_3 + x_4 \leqslant 56$ with non-negative integers x_1, x_2, x_3, x_4. (**Hint:** Introduce x_5.)
(e) $x_1 + x_2 + x_3 + x_4 \leqslant 56$ with integers $x_1 \geqslant 1$, $x_2 \geqslant 2$, $x_3 \geqslant 3$, $x_4 \geqslant 4$.
(f) $x_1+x_2+x_3+x_4 \geqslant 56$ with integers $x_1 \leqslant 11$, $x_2 \leqslant 22$, $x_3 \leqslant 33$, $x_4 \leqslant 44$. (**Hint:** Take $y_i := 11i - x_i$.)

12. **(a)** Let A and B finite sets of sizes n and m respectively. Use the principle of inclusion and exclusion to deduce that the number of surjective functions $A \to B$ equals

$$m^n - \binom{m}{1}(m-1)^n + \binom{m}{2}(m-2)^n - \cdots + (-1)^{m-1}\binom{m}{m-1}1^n .$$

Conclude that the Stirling number $S(n,m)$ equals $\frac{1}{m!}\sum_{i=0}^{m-1}(-1)^i\binom{m}{i}(m-i)^n$.

(b) In how many ways can six persons occupy three rooms so that no room remains vacant?

*** 13.** The principle of inclusion and exclusion is often stated in the following form. Prove it.

Let X be a set, $\mathcal{P}(X)$ the power set of X, and let $f, g : \mathcal{P}(X) \to \mathbb{R}$ be functions such that $f(A) = \sum_{S \subseteq A} g(S)$ for all subsets A of X. Then $g(A) = \sum_{S \subseteq A} (-1)^{|A|-|S|} f(S)$ for all subsets A of X.

14. Let $S(n,r)$ denote the Stirling numbers of the second kind and B_n the Bell numbers. Prove that:

(a) $S(n,2) = 2^{n-1} - 1$ for all $n \geqslant 2$.
(b) $S(n,n-1) = \binom{n}{2}$ for all $n \geqslant 2$.
(c) $x^n = \sum_{r=0}^{n} S(n,r) x^{\underline{r}}$, where $x^{\underline{r}} = x(x-1)\cdots(x-r+1)$.
(d) $S(n,r) = \sum_{k=r}^{n} r^{n-k} S(k-1, r-1)$.
(e) $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$.

15. A *permutation* π of $1, 2, \ldots, n$ is a rearrangement $\pi_1, \pi_2 \ldots, \pi_n$ of these elements. For example, $4, 6, 1, 3, 5, 2$ is a permutation of $1, 2, 3, 4, 5, 6$. We may view π as a bijective function from $\{1, 2, \ldots, n\}$ to itself, where $\pi(i) = \pi_i$. For example, the permutation $4, 6, 1, 3, 5, 2$ of $1, 2, 3, 4, 5, 6$ is viewed as:

i	1	2	3	4	5	6
$\pi(i)$	4	6	1	3	5	2

If $\pi(a_1) = a_2, \pi(a_2) = a_3, \ldots, \pi(a_{k-1}) = a_k, \pi(a_k) = a_1$, we say that $a_1, a_2, \ldots, a_k$ form a *cycle* of length k, denoted $(a_1, a_2, \ldots, a_k)$. Each permutation can be decomposed into pairwise disjoint cycles. For example, the permutation $4, 6, 1, 3, 5, 2$ has three cycles $(1, 4, 3), (2, 6), (5)$. The number of permutations of $1, 2, \ldots, n$ having exactly r cycles is called the *Stirling number of the first kind* and is denoted by $s(n,r)$. Prove the following identities:

(a) $s(n,r) = 0$ if $r > n$.
(b) $s(n,0) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \geqslant 0. \end{cases}$
(c) $s(n,1) = (n-1)!$ for all $n \geqslant 1$.
(d) $s(n,n-1) = \binom{n}{2}$ for all $n \geqslant 2$.
(e) $s(n,n) = 1$ for all $n \geqslant 0$.
(f) $\sum_{r=0}^{n} s(n,r) = n!$ for all $n \geqslant 0$.
(g) $s(n,r) = s(n-1,r-1) + (n-1)s(n-1,r)$ for all $n \geqslant 1$.
(h) $x^{\overline{n}} = \sum_{r=0}^{n} s(n,r) x^r$, where $x^{\overline{n}} = x(x+1)\cdots(x+n-1)$.

16. Define a recurrence relation and the requisite number of initial conditions for each of the following:

(a) The number of binary strings of length n, that do not contain three consecutive 0's.

* **(b)** The number of binary strings of length n, that do not contain k consecutive 0's, where $k \in \mathbb{N}_0$ is a constant. (**Hint:** Look at the first occurrence of 1.)

17. How many initial conditions are needed for completely and uniquely specifying the sequences defined by the following recurrence relations? Also explicitly mention for which values of n one should specify these initial conditions. Assume that each sequence a_n in this exercise starts from $n = 0$.

(a) $a_n = a_{n-2}$.
(b) $a_n = a_2$.
(c) $a_n = 2a_{n-2}^2 + 3a_{n-3}^3 + 4^{n-4}$.
(d) $a_n = (n-1)a_{n-1} + (n-2)a_{n-2} + \cdots + 2a_2 + a_1$.
(e) $a_n = na_{n-1} + (n-1)a_{n-2} + \cdots + 3a_2 + 2a_1 + a_0$.
(f) $a_n = a_{\lfloor n/2 \rfloor} a_{\lceil n/2 \rceil} + 1$.

18. Solve the following recurrence relations:

(a) $a_0 = 2$, $a_1 = 3$, $a_n = a_{n-1} + 12a_{n-2}$ for $n \geqslant 2$.
(b) $a_0 = 2$, $a_1 = 3$, $6a_n = a_{n-1} + 12a_{n-2}$ for $n \geqslant 2$.
(c) $a_0 = 2$, $a_1 = 3$, $a_2 = 4$, $a_n = a_{n-1} + 4a_{n-2} - 4a_{n-2}$ for $n \geqslant 3$.
(d) $a_0 = 2$, $a_1 = 3$, $a_2 = 4$, $a_3 = 5$, $a_n = 2a_{n-2} - a_{n-4}$ for $n \geqslant 4$.
(e) $a_0 = 2$, $a_1 = 3$, $a_2 = 4$, $a_3 = 5$, $a_n = 3a_{n-2} - 2a_{n-4}$ for $n \geqslant 4$.
(f) $a_0 = 2$, $a_n = 5a_{n-1} + (n^2 + n + 1)$ for $n \geqslant 1$.
(g) $a_0 = 2$, $a_n = 5a_{n-1} + (n^2 + n + 1)2^n$ for $n \geqslant 1$.
(h) $a_0 = 2$, $a_1 = 3$, $a_n = 2(a_{n-1} + a_{n-2} + 2^n)$ for $n \geqslant 2$.
(i) $a_0 = 2$, $a_1 = 3$, $a_n = 4(a_{n-1} - a_{n-2} + 2^n)$ for $n \geqslant 2$.
(j) $a_0 = 2$, $a_1 = 3$, $a_n = 4(a_{n-1} - a_{n-2} + n2^{2n-1})$ for $n \geqslant 2$.

19. Solve the following recurrence relations:

(a) $a_0 = 2$, $a_n = 2a_{n-1} + 2^n + n^2$ for all $n \geqslant 1$.
(b) $a_0 = 2$, $a_1 = 3$, $a_n = 2a_{n-1} - a_{n-2} + 2^n + n^2$ for all $n \geqslant 2$.
(c) $a_0 = 2$, $a_1 = 3$, $a_n = a_{n-2} + 2^n + n3^n + n^2 4^n$ for all $n \geqslant 2$.

20. Reduce the following recurrence relations to standard forms and solve:

(a) $a_0 = 2$, $a_1 = 3$, $a_n = a_{n+2} - a_{n+1} - n$ for all $n \geqslant 0$.
(b) $a_0 = 2$, $a_1 = 3$, $4a_{n+1} + 8a_n - 5a_{n-1} = 2^n$ for all $n \geqslant 1$.
(c) $a_0 = 2$, $a_1 = 3$, $a_n = a_{n-1} + 12(a_{n-2} + 2^{n-2} + 1)$ for all $n \geqslant 2$.
(d) $a_0 = 2$, $a_n^3 = a_{n-1}(3a_n^2 - 3a_n a_{n-1} + a_{n-1}^2) + n^3$ for all $n \geqslant 1$.
(e) $a_0 = 2$, $a_1 = 3$, $2a_n a_{n-2} - 2a_{n-1}^2 - 3a_{n-1}a_{n-2} = 0$ for all $n \geqslant 2$.
(f) $a_0 = 2$, $a_1 = 3$, $2^{a_n} = 4^n \times 16^{a_{n-2}}$ for all $n \geqslant 2$.

21. Find big-Oh estimates for the following positive-integer-valued increasing functions $f(n)$.

(a) $f(n) = 125f(n/4) + 2n^3$ whenever $n = 4^k$ for $k \in \mathbb{Z}^+$.
(b) $f(n) = 125f(n/5) + 2n^3$ whenever $n = 5^k$ for $k \in \mathbb{Z}^+$.
(c) $f(n) = 125f(n/6) + 2n^3$ whenever $n = 6^k$ for $k \in \mathbb{Z}^+$.

22. Let $f(n)$ be an increasing positive-real-valued function of a non-negative integer variable n. Give a big-Oh estimate of $f(n)$ for each of the following cases:

(a) $f(n) = 2f(\sqrt{n}) + 1$ whenever n is a perfect square bigger than 1.
(b) $f(n) = 2f(\sqrt{n}) + \log n$ whenever n is a perfect square bigger than 1.
(c) $f(n) = 2f(\sqrt{n}) + \log^2 n$ whenever n is a perfect square bigger than 1.
(d) $f(n) = af(\sqrt[b]{n}) + c(\log n)^d$ whenever n is a perfect b-th power bigger than 1. Here $a, b \in \mathbb{N}$, $a \geqslant 1$, $b \geqslant 2$, $c, d \in \mathbb{R}$, $c > 0$ and $d \geqslant 0$.

23. Find the generating functions for the following sequences:

(a) $1^3, 2^3, 3^3, \ldots, (n+1)^3, \ldots$ **(b)** $1^4, 2^4, 3^4, \ldots, (n+1)^4, \ldots$

(c) $\frac{1}{2}, \frac{1 \times 3}{2 \times 4}, \frac{1 \times 3 \times 5}{2 \times 4 \times 6}, \ldots, \frac{1 \times 3 \times 5 \times \cdots \times (2n+1)}{2 \times 4 \times 6 \times \cdots \times (2n+2)}, \ldots$

24. Use generating functions to solve the following recurrence relations:

(a) $a_0 = 2$, $a_n = 5a_{n-1} + 4n + 3$ for $n \geqslant 1$.
(b) $a_0 = 2$, $a_1 = 3$, $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geqslant 2$.
(c) $a_0 = 2$, $a_1 = 3$, $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$ for $n \geqslant 2$.
(d) $a_0 = 2$, $a_1 = 3$, $a_n = 3a_{n-1} - 2a_{n-2} + 2^n$ for $n \geqslant 2$.
(e) $a_0 = 2$, $a_1 = 3$, $a_n = 3a_{n-1} - 2a_{n-2} + 2^n + 1$ for $n \geqslant 2$.
(f) $a_0 = 2$, $a_1 = 3$, $a_n = 4a_{n-1} - 4a_{n-2} + 2^n + 1$ for $n \geqslant 2$.

25. The n-th Catalan number C_n stands for the number of ways of fully parenthesizing the product $x_0 x_1 \cdots x_n$. Prove that Catalan numbers satisfy the recurrence:

$$\begin{aligned} C_0 &= 1 \\ C_n &= \sum_{i=0}^{n-1} C_i C_{n-i-1} \text{ for } n \geqslant 1. \end{aligned}$$

Define the power series $C(x) = C_0 + C_1 x + C_2 x^2 + \cdots = \sum_{n \in \mathbb{N}_0} C_n x^n$. Show that this power series satisfies $C(x) = 1 + xC(x)^2$, so that $C(x) = \frac{1 - \sqrt{1-4x}}{2x}$. Conclude that $C_n = \frac{1}{n+1}\binom{2n}{n}$.

*** 26.** Catalan numbers are known to have a bunch of combinatorial interpretations. Prove that C_n equals each of the following:

(a) The number of sequences $a_1 a_2 \ldots a_{2n}$ of length $2n$ with each $a_i \in \{1, -1\}$, $\sum_{i=1}^{2n} a_i = 0$, and $\sum_{i=1}^{j} a_i \geqslant 0$ for all $j \in \{1, 2, \ldots, 2n\}$.
(b) The number of strings consisting of $2n$ balanced left and right parentheses.
(c) The number of ways of drawing n non-intersecting chords with $2n$ distinct endpoints on the circumference of a circle.
(d) The number of (rooted) binary trees with n internal vertices. (An internal vertex is one that has at least one child.)
(e) The number of paths from the lower left corner to the top right corner in an $n \times n$ grid that do not rise above the grid diagonal connecting the two corners mentioned above.
(f) The number of ways an $(n+2)$-gon can be cut into triangles.

27. Let $a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$, $b(x) = b_0 + b_1 x + b_2 x^2 + \cdots$ and $c(x) = c_0 + c_1 x + c_2 x^2 + \cdots$ be (formal) power series. Show that:

(a) $a(x)(b(x) + c(x)) = a(x)b(x) + a(x)c(x)$.

(b) $a(x)$ is invertible (i.e., there exists a power series $d(x)$ with $a(x)d(x) = 1$) if and only if $a_0 \neq 0$.