CS21001 Discrete Structures, Autumn 2005

Exercise set 2

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1. Determine which of the following functions are injective, which are surjective, and which are bijective. Also describe the inverses of the bijective functions.

(a) The function $\mathbb{Z} \to \mathbb{N}_0$ that maps $n \mapsto |n|$.
(b) The function $\mathbb{Z} \to \mathbb{N}$ that maps $n \mapsto 2n+1$ for $n \geqslant 0$ and $-n \mapsto 2n+2$ for $n > 0$.
(c) The function $f : \mathbb{N} \to \mathbb{N}$ with $f(1) = 9$ and $f(n+1) = \begin{cases} 3f(n)+1 & \text{if } f(n) \text{ is odd} \\ \frac{1}{2}f(n) & \text{if } f(n) \text{ is even} \end{cases}$ for $n \geqslant 1$.
(d) A function $f : A \to A$ satisfying $f(f(x)) = x$ for all $x \in A$.
(e) A function $f : A \to A$ satisfying $f(f(f(x))) = x$ for all $x \in A$.
(f) A function $f : A \to A$ satisfying $f(f(x)) = f(x)$ for all $x \in A$.
(g) A function $f : A \to A$ satisfying $f(f(f(x))) = f(x)$ for all $x \in A$.

2. Construct explicit bijections between $\mathbb{N}$ and the following sets:

(a) $\mathbb{N} \times \mathbb{N}$.
(b) The subset of $\mathbb{N}$ comprising integers whose decimal representations consist of the digit 5 only.
(c) The subset of $\mathbb{N}$ comprising integers whose decimal representations consist of the digits $4, 5$ only.
(d) The subset of $\mathbb{N}$ comprising integers whose decimal representations consist of the digits $4, 5, 6$ only.

3. Let A, B be finite sets with $|A| = m$ and $|B| = n$. Determine the numbers of:

(a) functions $A \to B$,
(b) injective functions $A \to B$ (provided that $m \leqslant n$),
** **(c)** surjective functions $A \to B$ (provided that $m \geqslant n$),
(d) bijective functions $A \to B$ (provided that $m = n$),
(e) symmetric relations on A,
(f) reflexive and symmetric relations on A.

4. Let $f : A \to B$ and $g : B \to C$ be functions.

(a) If f and g are both injective, prove that $g \circ f$ is injective too.
(b) If f and g are both surjective, prove that $g \circ f$ is surjective too.
(c) If f and g are both bijective, prove that $g \circ f$ is bijective too. Show also that in this case we have $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
(d) If $g \circ f$ is injective, prove that f is injective too.
(e) If $g \circ f$ is surjective, prove that g is surjective too.
(f) Give an example in which g is not injective, but $g \circ f$ is injective.
(g) Give an example in which f is not surjective, but $g \circ f$ is surjective.

5. Let A be a set of cardinality $n = |A|$ and $f : A \to A$ a function.

(a) If n is finite, show that f is injective if and only if f is surjective.

(b) If $n = \infty$, demonstrate by examples that neither of the implications of Part (a) is true.

6. Let $f : A \to B$ be a function. For $S \subseteq A$ define the set $f(S) = \{f(x) \mid x \in S\} \subseteq B$. Also for $T \subseteq B$ define $f^{-1}(T) = \{x \in A \mid f(x) \in T\} \subseteq A$. Let S, S_1, S_2 be subsets of A and T, T_1, T_2 subsets of B. Prove the following assertions:

(a) If $S_1 \subseteq S_2$, then $f(S_1) \subseteq f(S_2)$.
(b) If $T_1 \subseteq T_2$, then $f^{-1}(T_1) \subseteq f^{-1}(T_2)$.
(c) $S \subseteq f^{-1}(f(S))$.
(d) $f(f^{-1}(T)) \subseteq T$.
(e) $f(f^{-1}(f(S))) = f(S)$.
(f) $f^{-1}(f(f^{-1}(T))) = f^{-1}(T)$.

*** 7.** Let A be a set and $\mathcal{P}(A)$ the power set of A. Prove that there cannot exist a bijection between A and $\mathcal{P}(A)$.

*** 8.** Let $f : A \to A$ be a bijection. Assume that there exists a positive integer r such that the r-fold composition f^r of f is the identity map ι_A. The smallest such (positive) r is called the *order* of f. If no such r exists, we say that f is of *infinite order*.

Take $A = \mathbb{Z}$ and consider the following functions $\mathbb{Z} \to \mathbb{Z}$:

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even,} \\ n-1 & \text{if } n \text{ is odd.} \end{cases}$$
$$g(n) = \begin{cases} n-1 & \text{if } n \text{ is even,} \\ n+1 & \text{if } n \text{ is odd.} \end{cases}$$

Prove the following assertions:

(a) Both f and g are bijections on $\mathbb{Z}$.
(b) Both f and g are of order 2.
(c) $g \circ f$ is of infinite order.

9. Which of the following relations are reflexive? Symmetric? Anti-symmetric? Transitive?

(a) $\subseteq$ on $\mathcal{P}(A)$, where A is a non-empty set.
(b) The relation R_1 on $\mathbb{N}$ defined as: $a\,R_1\,b$ if and only if the decimal representations of a and b start with the same digit.
(c) The relation R_2 on $\mathbb{N}$ defined as: $a\,R_2\,b$ if and only if the decimal representations of a and b contain a common digit. (Do not allow leading 0 digits.)
(d) The relation R_3 on $\mathbb{N} \times \mathbb{N}$ defined as: $(a,b)\,R_3\,(c,d)$ if and only if ($a \leqslant b$) or ($a = b$ and $c \leqslant d$).

10. Take some fixed $n \in \mathbb{N}_0$ and let $\mathcal{M}$ denote the set of all $n \times n$ matrices with real entries. Two matrices $A, B \in \mathcal{M}$ are called *similar* if $B = PAP^{-1}$ for some invertible (i.e., nonsingular) matrix $P \in \mathcal{M}$. Verify that similarity is an equivalence relation on $\mathcal{M}$.

11. Let A denote the set of all functions $\mathbb{N} \to \mathbb{N}$. Define a relation $\preceq$ on A as follows: $f \preceq g$ if and only if $f(n) \leqslant g(n)$ for all $n \in \mathbb{N}$. Argue that $\preceq$ is a partial order on A.

12. Let A denote the set of all positive real-valued functions of natural numbers (or non-negative integers). Define a relation O on A as: $f\ \mathrm{O}\ g$ if and only if $f = \mathrm{O}(g)$. Also define the relation Θ on A as: $f\,\Theta\,g$ if and only if $f = \Theta(g)$.

(a) Prove that O is *not* a partial order on A.

(b) Prove that Θ is an equivalence relation on A.

(c) Denote by A' the set of equivalence classes of A with respect to Θ. Let $[f], [g] \in A'$ for some $f, g \in A$. Define a relation O' on A' as: $[f]\ \mathrm{O}'\ [g]$ if and only if $f = \mathrm{O}(g)$. Show that the relation O' is well-defined, i.e., independent of the choices of the representatives of $[f]$ and $[g]$. Prove that O' is a partial order on A'.

13. Prove the following assertions:

(a) Any infinite subset of a countable set is countable.
(b) If A and B are countable (but not necessarily disjoint), then $A \cup B$ and $A \times B$ are countable.
(c) The union of a countable collection of finite/countable sets is countable.
(d) The set of strings (finite sequences) of the symbols $0, 1$ is countable.
(e) The set $\mathbb{A} = \{\alpha \in \mathbb{C} \mid \alpha \text{ is a root of a non-zero polynomial with integer coefficients}\}$ is countable. (Elements of $\mathbb{A}$ are called *algebraic numbers*.)
(f) The set $\mathbb{C}$ of complex numbers is uncountable.
(g) The set of transcendental (i.e., non-algebraic) real numbers is uncountable.