CS21001 Discrete Structures, Autumn 2005

Exercise set 1

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1. Let A, B, C be sets. Which of the following statements are true?

(a) $A \setminus B = A \cap \bar{B}$.

(b) $(A \setminus B) \setminus C = A \setminus (B \setminus C)$.

(c) $A \setminus (B \cup C) = (A \setminus B) \cup (A \setminus C)$.

(d) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

2. Let p, q, r be propositions. Which of the following statements are true?

(a) $p \Rightarrow q$ is equivalent to $\neg p \vee q$.

(b) $(p \Rightarrow q) \wedge (q \Rightarrow r)$ is equivalent to $p \Rightarrow r$.

(c) $(p \Rightarrow q) \wedge (q \Rightarrow r)$ is equivalent to $(p \Rightarrow q) \wedge (p \Rightarrow r)$.

(d) $p \Rightarrow (q \Rightarrow r)$ is equivalent to $(p \wedge q) \Rightarrow r$.

(e) $\neg p \Rightarrow (q \Rightarrow r)$ is equivalent to $q \Rightarrow (p \vee r)$.

3. Which of the following statements are true?

(a) $\forall x \in \mathbb{N}\, \exists y \in \mathbb{N} \left[(y > 1) \wedge (y^2 \mid x) \right]$.

(b) $\forall x \in \mathbb{N}\, \exists y \in \mathbb{N}\, \forall z \in \mathbb{N} \setminus \{1\} \left[(y > x) \wedge (z^2 \mid y) \right]$.

(c) $\forall x \in \mathbb{N}\, \forall z \in \mathbb{N} \setminus \{1\}\, \exists y \in \mathbb{N} \left[(y > x) \wedge (z^2 \mid y) \right]$.

(d) For all integers $n > 1$, $n^4 + 4^n$ is composite.

(e) There exist positive integers x, y with $x \neq y$ such that $x^y = y^x$.

**** (f)** There exist irrational numbers x, y such that x^y is rational.

4. Consider the statement:

u : If n and $n^2 + 8$ are prime, then $n^3 + 4$ and $n^4 + 2$ are prime.

(a) What is the converse statement of u?

(b) What is the contrapositive statement of u?

*** (c)** Prove that u is true for all $n \in \mathbb{N}$.

*** 5.** Let $f(n) = a_d n^d + a_{d-1} n^{d-1} + \cdots + a_1 n + a_0$, $d > 0$, $a_d > 0$, be a polynomial with integer coefficients. Prove that there exists an integer n for which $f(n)$ is composite.

6. Use the principle of mathematical induction to prove the following assertions:

(a) $2^n > n$ for all $n \in \mathbb{N}$.

(b) $2^n \geqslant n^2$ for all integer $n \geqslant 4$.

(c) $1^4 + 2^4 + \cdots + n^4 = \frac{1}{30} n(n+1)(2n+1)(3n^2 + 3n - 1)$.

(d) Let a_n, $n \in \mathbb{N}$, be a sequence satisfying the following: $a_1 = 2$, $a_2 = 10$, and $a_n = 2a_{n-1} + 3a_{n-2}$ for all $n \in \mathbb{N}$. Then $a_n > 3^{n-1}$ for all $n \in \mathbb{N}$.

(e) The sequence a_n, $n \in \mathbb{N}$, of Part (d) has the closed form formula: $a_n = 3^n + (-1)^n$ for all $n \in \mathbb{N}$.

(f) $x^{2n+1} + y^{2n+1}$ is divisible by $x + y$ for all integer $n \geqslant 0$.

(g) $1/\sqrt{1} + 1/\sqrt{2} + \cdots + 1/\sqrt{n} > 2(\sqrt{n+1} - 1)$ for all $n \in \mathbb{N}$.

(h) $H_1 + H_2 + \cdots + H_n = (n+1)H_n - n$ for all $n \in \mathbb{N}_0$, where H_n denotes the n-th harmonic number.

(i) $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$ for all $n \geqslant 0$, where F_n denotes the n-th Fibonacci number.

(j) $F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$ for all $n \geqslant 0$.

(k) If $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, then $A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$ for all $n \in \mathbb{N}$.

(l) $F_{m+n} = F_{m-1}F_n + F_m F_{n+1}$ for all $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$.

7. Find the flaw in the following proof:

Theorem: All horses are of the same color.

Proof Let there be n horses. We proceed by induction on n. If $n = 1$, there is nothing to prove. So assume that $n > 1$ and that the theorem holds for any group of $n - 1$ horses. From the given n horses discard one, say the first one. Then all the remaining $n - 1$ horses are of the same color by the induction hypothesis. Now put the first horse back and discard another, say the last one. Then the first $n - 1$ horses have the same color again by the induction hypothesis. So all the n horses must have the same color as the ones that were not discarded either time. •

8. What is the problem in the following inductive definition of sets.

Basis: A collection with no members is a set.

Induction: If S is a set and x is an object not present in S, then the collection obtained by adding x to S is also a set.

9. Inductively define the following sequences a_n, $n \in \mathbb{N}$.

(a) $a_n = 2n^3$.

(b) $a_n = n^2 + 2n^3$.

(c) $a_n = 3^{2n}$.

(d) $a_n = 3^{2^n}$.

* **(e)** $a_n = 2^n + 3^{2n}$.

10. **(a)** Let a_n denote the number of strings of length n over the lower-case Roman alphabet $\{a, b, c, \ldots, z\}$ containing two consecutive vowels. Find a recursive formula for a_n.

(b) Let b_n denote the number of strings of length n over the lower-case Roman alphabet $\{a, b, c, \ldots, z\}$ *not* containing two consecutive consonants. Find a recursive formula for b_n.