CS21001 Discrete Structures, Autumn 2005

Class test 1 : Solutions

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1. Which of the following assertions is/are true. Give one-line justifications. No credits will be given without proper reasoning. **(2×5)**

(a) The proposition $\neg p \Rightarrow q \vee r$ is equivalent to the proposition $\neg q \Rightarrow p \vee r$.

True: $\neg p \Rightarrow q \vee r \equiv \neg(\neg p) \vee q \vee r \equiv p \vee q \vee r \equiv q \vee p \vee r \equiv \neg(\neg q) \vee p \vee r \equiv \neg q \Rightarrow p \vee r$.

(b) The function $f : \mathbb{Q} \to \mathbb{N}$ that maps a/b with $\gcd(a, b) = 1$ to $a^2 + b^2$ is injective.

False: $f(2/1) = f(1/2) = 5$.

(c) Let $g : \mathbb{Z} \to \mathbb{Z}$ be a function satisfying $g(a + b) = g(a) + g(b)$ for all $a, b \in \mathbb{Z}$. Then $g(0) = 0$.

True: $g(0) = g(0 + 0) = g(0) + g(0)$.

(d) Define a relation R on $\mathbb{N}$ as follows: Let $m, n \in \mathbb{N}$. Write $m = 2^s a$ and $n = 2^t b$ with a, b odd. Define $m\ R\ n$ if and only if $s \leqslant t$. Then R is a partial order on $\mathbb{N}$.

False: $6\ R\ 10$ and $10\ R\ 6$, but $6 \neq 10$.

(e) If A and B are uncountable sets, then $A \cap B$ must be an uncountable set.

False: Take $A = \{x + \mathrm{i}0 \mid x \in \mathbb{R}\}$ and $B = \{0 + \mathrm{i}y \mid y \in \mathbb{R}\}$. Then $A \cap B = \{0 + \mathrm{i}0\}$ is finite.

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2. Define a relation ρ on $\mathbb{R}$ as $a\ \rho\ b$ if and only if $a - b \in \mathbb{Q}$.

(a) Prove that ρ is an equivalence relation on $\mathbb{R}$. **(5)**

[Reflexive] $0 = 0/1$ is a rational number.

[Symmetric] If $a - b$ is rational, then $b - a = -(a - b)$ is rational too.

[Transitive] If $a - b$ and $b - c$ are rational, their sum $a - c = (a - b) + (b - c)$ is again rational.

(b) Prove that the set $\mathbb{R}/\rho$ of equivalence classes of $\mathbb{R}$ under ρ is uncountable. **(5)**

Each equivalence class $[x]$ of ρ is of the form $[x] = \{x + r \mid r \in \mathbb{Q}\}$, i.e., each $[x]$ has a bijective correspondence with $\mathbb{Q}$ and so is countable. If the number of equivalence classes is countable too, then the union of all these classes would again be countable. But $\mathbb{R}$ is uncountable.

(c) [*Take-home bonus*] Describe an explicit bijection between the sets $\mathbb{R}$ and $\mathbb{R}/\rho$. **(10)**

3. Use a diagonalization argument to prove that the set of all functions $\mathbb{N} \to \mathbb{N}$ is uncountable. No credit will be given to proofs that are not based on diagonalization arguments. **(5)**

Let A be the set of all functions $\mathbb{N} \to \mathbb{N}$. Assume that A is countable and let $\varphi : \mathbb{N} \to A$ be a bijection. Denote the function $\varphi(n) : \mathbb{N} \to \mathbb{N}$ by φ_n. Define a function $g : \mathbb{N} \to \mathbb{N}$ as

$$g(n) = \varphi_n(n) + 1$$

for all $n \in \mathbb{N}$. Then g differs from each φ_n, since by construction $g(n) \neq \varphi_n(n)$. This implies that φ is not surjective, a contradiction.