Class Test II, Autumn 2003-04

Solutions

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1. We have $\mathrm{HALFCYCLE} = L_1 \setminus L_2$, where $L_1 := \{\langle G\rangle \mid \text{The digraph } G \text{ has a cycle of length} \geqslant \lfloor n(G)/2 \rfloor\}$ and $L_2 := \{\langle G\rangle \mid \text{The digraph } G \text{ has a cycle of length } > \lfloor n(G)/2 \rfloor\}$. Each of L_1 and L_2 is in NP, since an explicitly given sufficiently big cycle in G is a succinct certificate for the membership of $\langle G\rangle$ in the language. Thus $\mathrm{HALFCYCLE} \in \mathrm{DP}$.

2. $\Delta_1\mathrm{P} = \mathrm{P}^{\Sigma_0\mathrm{P}} = \mathrm{P}^{\mathrm{P}} = \mathrm{P}$.

Taking $L_2 = \emptyset$ in the (first) definition of DP shows that $\mathrm{NP} \subseteq \mathrm{DP}$. Moreover, taking $L_1 = \Sigma^*$ in this definition indicates that $\mathrm{coNP} \subseteq \mathrm{DP}$. Thus $\mathrm{NP} \cup \mathrm{coNP} \subseteq \mathrm{DP}$.

Finally, let $L \in \mathrm{DP}$. $L = L_1 \setminus L_2$ for some $L_1, L_2 \in \mathrm{NP}$. There exist poly-time reductions f_1 and f_2 from L_1 and L_2 to SAT. For an instance α for L, we ask the SAT oracle about $f_1(\alpha)$ and $f_2(\alpha)$ and accept if and only the first call returns 'yes' and the second 'no'. We conclude that $\mathrm{DP} \subseteq \mathrm{P}^{\mathrm{SAT}} = \mathrm{P}^{\mathrm{NP}} = \mathrm{P}^{\Sigma_1\mathrm{P}} = \Delta_2\mathrm{P}$.

3. [if] $\mathrm{NP} = \mathrm{coNP}$ implies that NP is closed under complementation. We also know that NP is closed under intersection. Since $L_1 \setminus L_2 = L_1 \cap \overline{L_2}$, it then follows that $\mathrm{DP} = \mathrm{NP} = \mathrm{coNP} = \mathrm{NP} \cup \mathrm{coNP}$.

[only if] I start by proving two auxiliary results:

Claim: $\mathrm{HAMCYCLE} \leqslant_P \mathrm{HALFCYCLE}$.

Let $\langle G\rangle$ be an instance for HAMCYCLE with $m := n(G)$. Call G' to be the digraph obtained by adding to G exactly m isolated vertices. It is clear that the longest cycle in G' is of length $m = \lfloor n(G')/2 \rfloor$, if and only if G contains a Hamiltonian cycle.

Claim: $\overline{\mathrm{HAMCYCLE}} \leqslant_P \mathrm{HALFCYCLE}$.

Let $\langle G\rangle$ be an instance for $\overline{\mathrm{HAMCYCLE}}$ with $m := n(G)$. Add $m-1$ new vertices to G and $m-1$ new edges, so that the new vertices form a directed cycle. Call the resulting graph G''. We have $n(G'') = 2m-1$ and so the longest cycle in G'' is of length $m - 1 = \lfloor n(G'')/2 \rfloor$, if and only if G does *not* contain a Hamiltonian cycle.

Now assume that $\mathrm{DP} = \mathrm{NP} \cup \mathrm{coNP}$. By Exercise 1 HALFCYCLE is either in NP or coNP. First consider that $\mathrm{HALFCYCLE} \in \mathrm{NP}$. By the second claim HALFCYCLE is coNP-hard. But we already know that if a coNP-hard language is in NP, then $\mathrm{NP} = \mathrm{coNP}$. So finally consider the case that $\mathrm{HALFCYCLE} \in \mathrm{coNP}$. By the first claim HALFCYCLE is NP-hard. Thus again we have $\mathrm{NP} = \mathrm{coNP}$.
