Class Test II, Autumn 2003-04

Total points: 10 | April 1, 2004 | Total time: 1 hour

The complexity class DP is defined as:

$$\mathrm{DP} := \{L_1 \setminus L_2 \mid L_1, L_2 \in \mathrm{NP}\} = \{L_1 \cap L_2 \mid L_1 \in \mathrm{NP}, L_2 \in \mathrm{coNP}\},$$

i.e., $L \in \mathrm{DP}$, if and only if L is the *difference* of two languages in NP, or equivalently, if and only if L is the intersection of a language in NP and a language in coNP.

The *difference hierarchy* is defined as

$$\begin{aligned} \Delta_0\mathrm{P} &:= \mathrm{P}, \\ \Delta_i\mathrm{P} &:= \mathrm{P}^{\Sigma_{i-1}\mathrm{P}} \text{ for } i \geqslant 1. \end{aligned}$$

1. Argue that the language

$$\mathrm{HALFCYCLE} := \{\langle G \rangle \mid \text{ The longest cycle in the directed graph } G \text{ is of length } \lfloor n(G)/2 \rfloor\}$$

is in DP. (Here $n(G)$ denotes the number of vertices of the graph G, and $\lfloor x \rfloor$ the floor of the real number x, i.e., the largest integer $\leqslant x$.) **(4)**

2. Demonstrate that $\mathrm{P} = \Delta_1\mathrm{P}$ and $\mathrm{NP} \cup \mathrm{coNP} \subseteq \mathrm{DP} \subseteq \Delta_2\mathrm{P}$. **(4)**

3. Prove that $\mathrm{DP} = \mathrm{NP} \cup \mathrm{coNP}$, if and only if $\mathrm{NP} = \mathrm{coNP}$. (**Hint:** For proving the 'only if' part you may first show that HALFCYCLE is both NP-hard and coNP-hard — use poly-time reductions from HAMCYCLE and $\overline{\mathrm{HAMCYCLE}}$.) **(2+5)**
