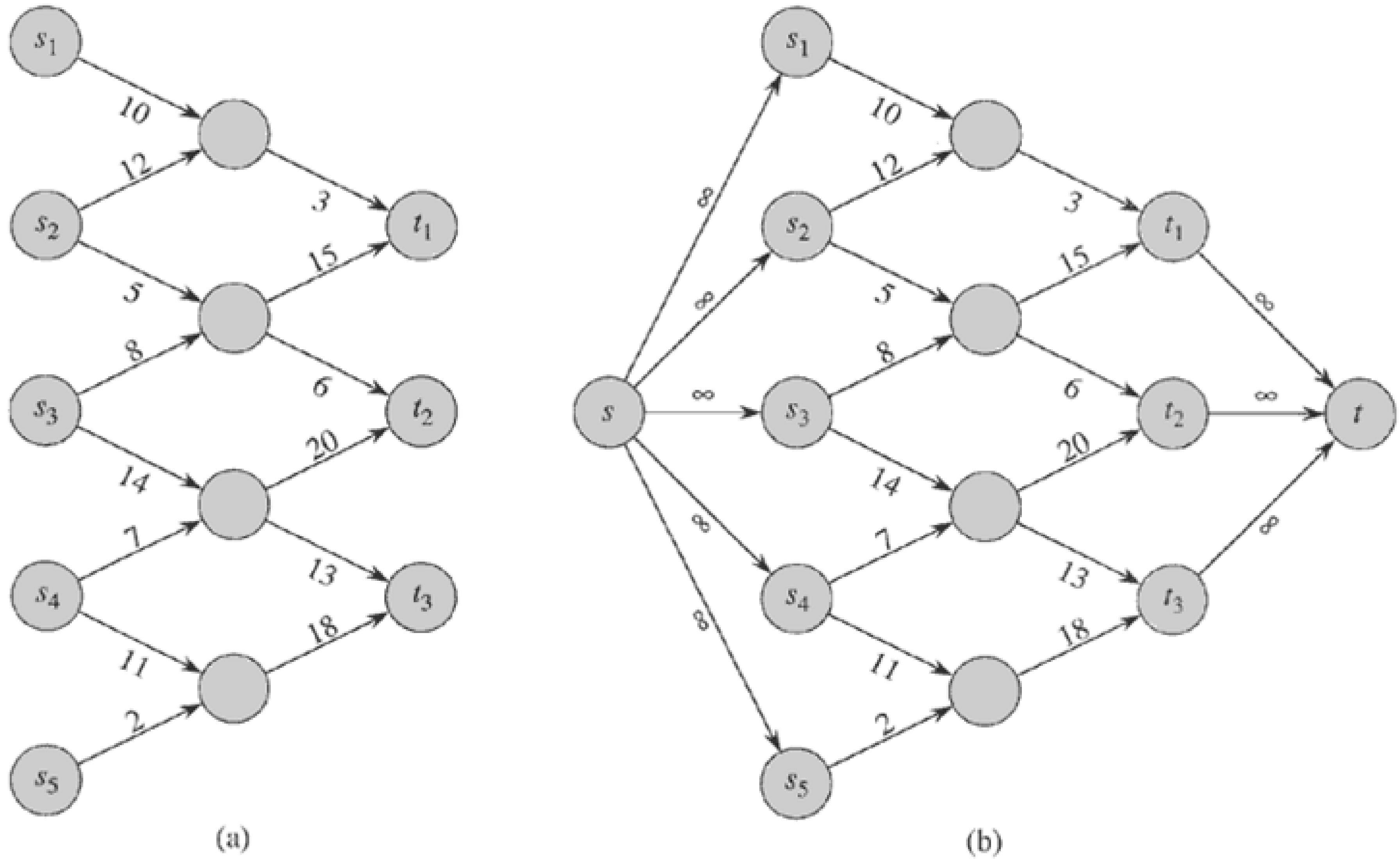
Edmonds-Karp Algorithm

- Proposed in 1972
- Almost same as Ford-Fulkerson
- Main difference: Uses BFS to find augmenting paths in residual graph instead of DFS
- You can prove that
  - If the Edmonds-Karp algorithm is run on a flow network $G = (V, E)$ with source s and sink t, then for all vertices $v \in V - \{s, t\}$, the shortest distance $\delta_f(s, v)$ in the residual network G_f increases monotonically with each flow augmentation
  - The total number of flow augmentations performed by the Edmonds-Karp algorithm is $O(VE)$
- This gives time complexity of Edmonds-Karp as $O(VE^2)$, as BFS can be done in $O(E)$

What if there are multiple sources and sink?

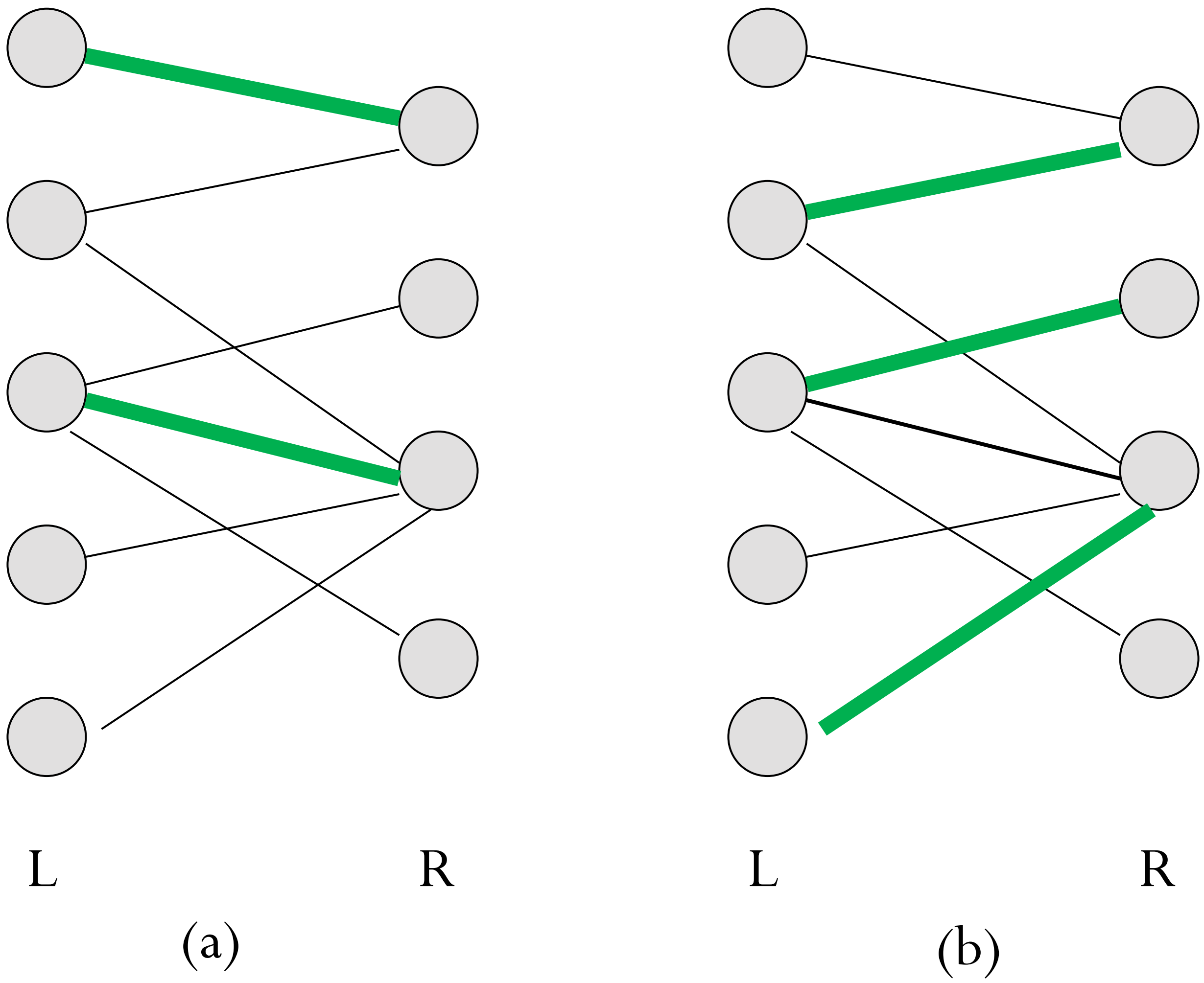
- Suppose there are multiple sources $s_1, s_2, s_3, \ldots s_p$ and multiple sinks $t_1, t_2, t_3, \ldots . t_q$
- How do we maximize the sum of the flows from all the sources to all the sinks?
- Can easily use the standard maximum flow problem
  - Add a "supersource" s with edge (s, s_j) from s to all sources s_j with capacity ∞
  - Add a "supersink" t with edge (t_j, t) from all sinks t_j to t with capacity ∞
  - Solve the maximum flow problem with s as source and t as sink



s_1
s_2
s_3
s_4
s_5
t_1
t_2
t_3
10
12
5
8
14
7
11
2
3
15
6
20
13
18
(a)
s
t
∞
(b)

Application: Maximum Cardinality Bipartite Matching

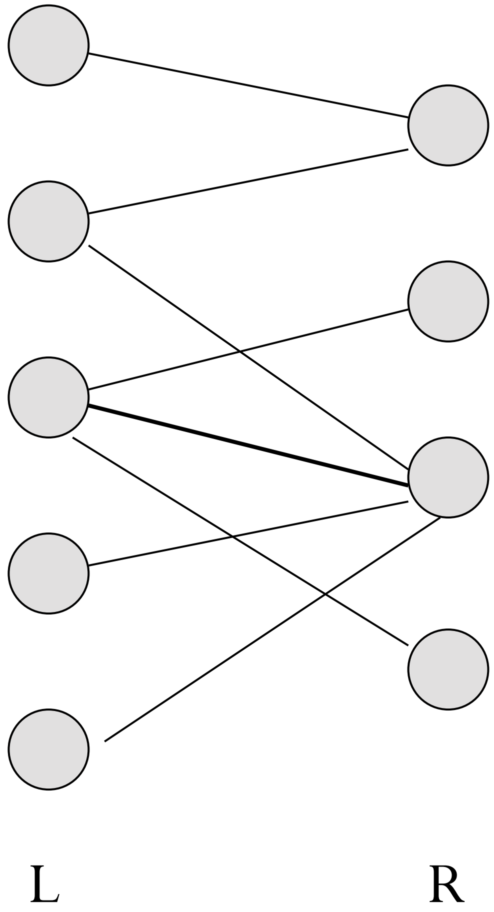
- Bipartite Graph: an undirected graph $G = (V, E)$ such that the vertex set can be partitioned $V = L \cup R$ where L and R are disjoint and there is no edge between two vertices in L or two vertices in R
- A matching in an undirected graph $G = (V, E)$ is a subset of edges $M \subseteq E$, such that for all vertices $v \in V$, at most one edge of M is incident on v.
- A maximum cardinality matching is a matching with maximum number of edges among all possible matchings
  - Also simply called maximum matching for unweighted graphs

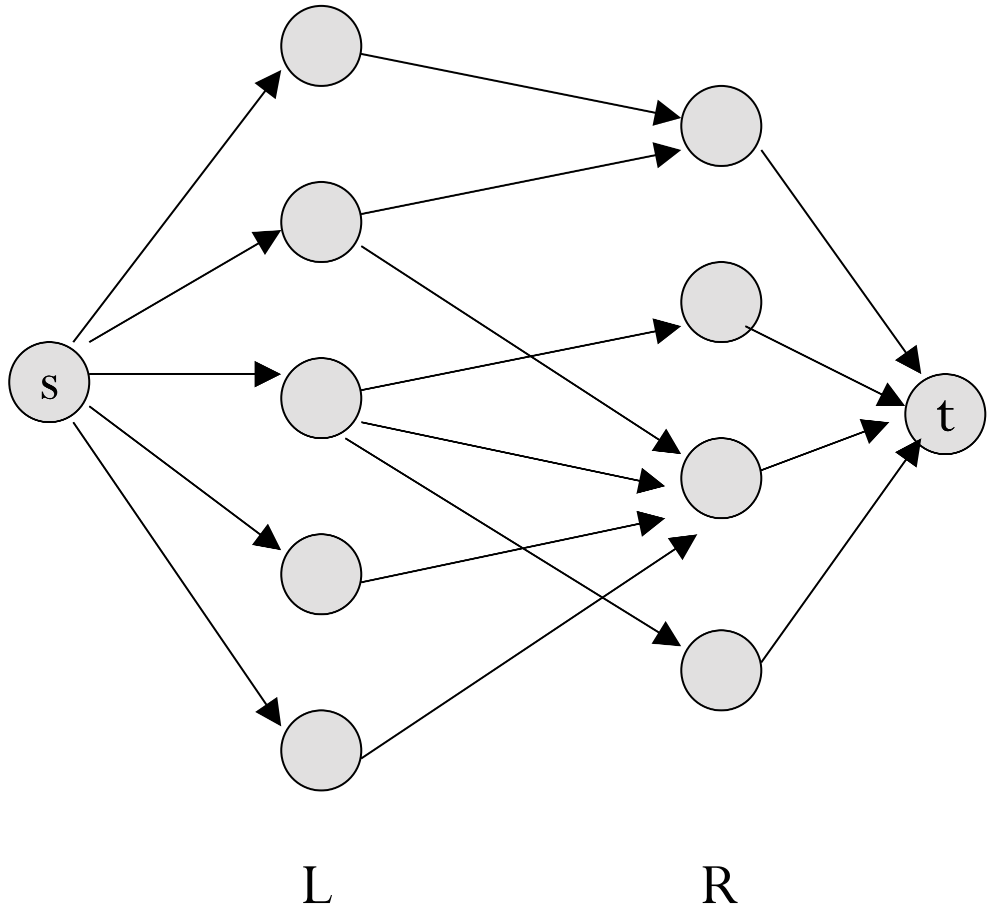


(a)A matching with cardinality 2

(b) A maximum matching with cardinality 3

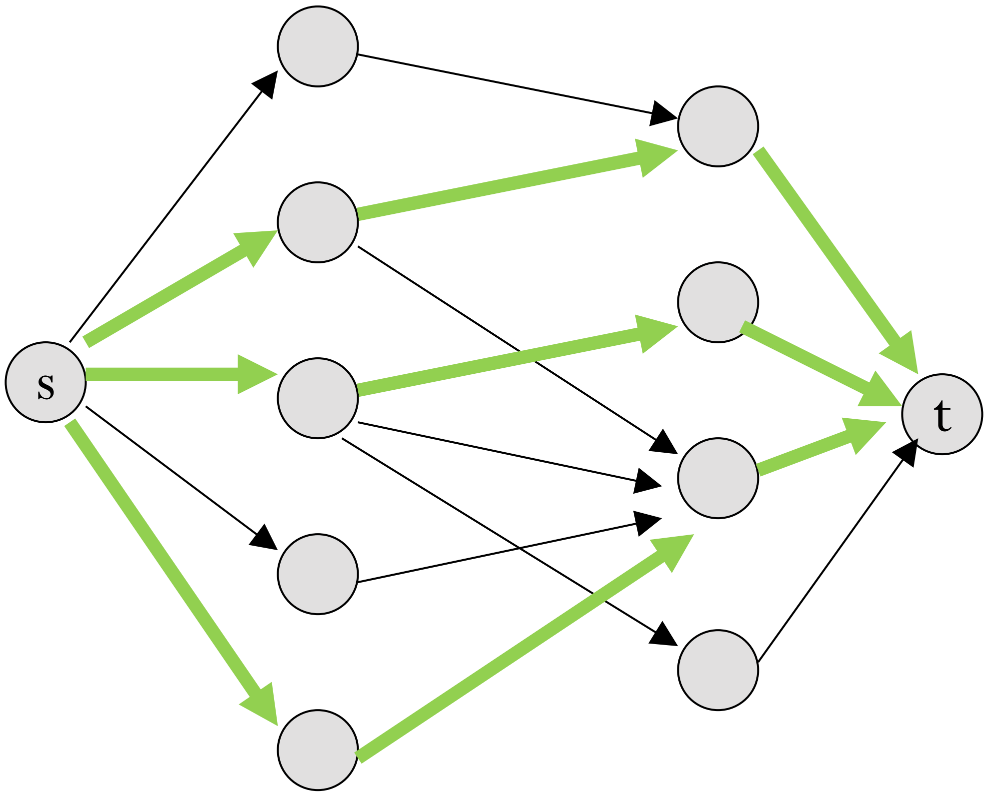
- Given the undirected bipartite graph G = (V, E) with partitions L and R, create a flow network G' = (V', E') as follows
  - Add two new vertices s, t. So V' = V ∪ {s, t}
  - For each node u in L, add a directed edge (s,u) with capacity 1 to E'
  - For each node v in R, add a directed edge (v,t) with capacity 1 to E'
  - For each edge (u,v) in E with u in L and v in R, add a directed edge (u, v) with capacity 1 to E'



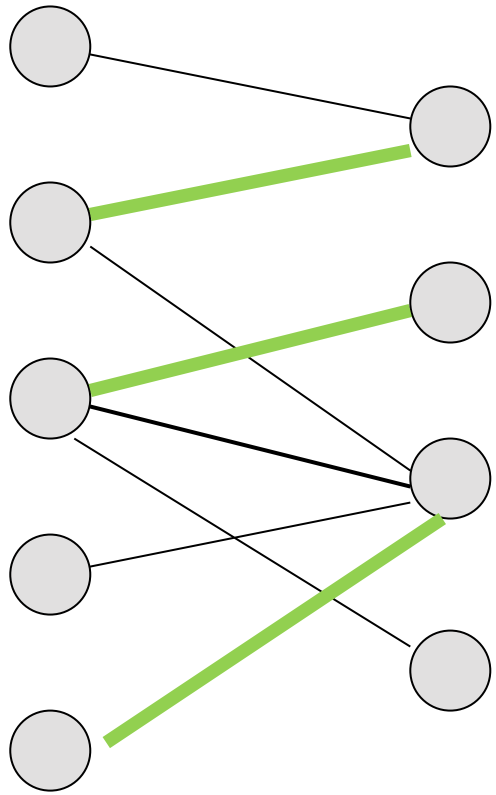


All capacities are 1

- Now solve the maximum flow problem from s to t in G'
- The edges of G with corresponding edges in G' with flow = 1 correspond to the maximum matching



Maximum flow found



Corresponding Maximum Matching

Application: Edge Connectivity

- Given an undirected graph $G = (V, E)$, edge connectivity of G is the minimum number of edges that have to be removed to disconnect the graph
  - A graph is called k-edge-connected if its edge connectivity is at least k
- Problem: Find the edge connectivity of a given undirected graph
- Important practical problem in various forms for different types of network design
  - Example: to avoid disruption in a computer network, need to ensure that a small number of link failures cannot disconnect the network

- We will use the maximum flow problem
- We know that the maximum flow is equal to the capacity of the minimum (S,T) cut
- So if we set all capacities to 1, the maximum flow value gives the minimum number of edges that goes across any cut (S,T), and so, the minimum number of edges that needs to be removed so that there is no path from s to t
- But the flow network is a directed graph, we need to solve it for an undirected graph
  - Easy. Maximum flow algorithms work on undirected graphs simply by converting it first to a directed graph, with each undirected edge replaced by two directed edges

- We also need to consider disconnection of any two vertices, not just two specified ones like s and t
  - So (u,v)-cuts for any two vertices u and v
  - Simple solution:
    - For each pair of vertices (u,v), set s=u, t=v and find the minimum cut size by solving the maximum flow problem
    - Take the minimum over all (u,v) pairs
    - Time complexity = no. of distinct pairs × max-flow time

      $= O(|V|^2) \times O(|V||E|^2)$ (using Edmonds-Karp)

      $= O(|V|^3|E|^2)$
    - Can do better, no need to consider all pairs

Input: Connected graph G = (V, E)

choose any vertex p in V

min_size = |E|

for all vertices $q \neq p$ do

find maxflow M in directed graph G' = (V, E')

where E' = { (u,v), (v,u) | (u,v) in E }

s = p, t = q, and all capacities = 1

min_size = min (min_size, M)

edge connectivity of G = min_size

Why is it sufficient to just find edge-connnectivity between a fixed *p* and all other vertices (and not between all pairs of vertices)?

Time Complexity = $(|V|^2|E|^2)$ (using Edmonds-Karp)

Preflow-Push Method

- Also called Push-Relabel method as it is based on two basic operations, push and relabel
- Main difference from Ford-Fulkerson based algorithms
  - Do not need to maintain the flow-conservation property throughout the execution
    - Total inflow at a vertex can be greater than total outflow from it in intermediate steps
    - But in the final solution, they must be the same as before

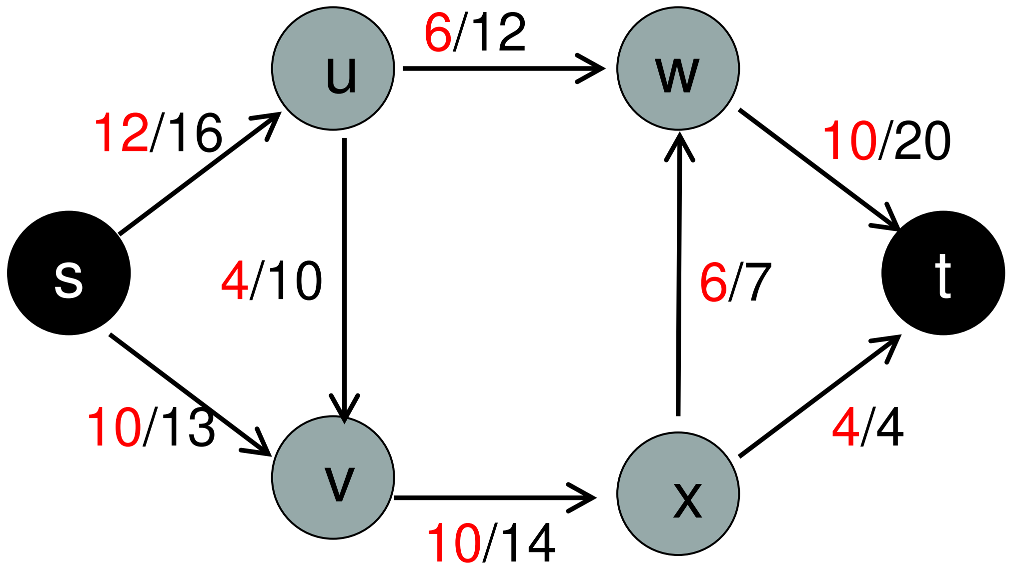
- Constraints satisfied by $f : V \times V \rightarrow R$ in intermediate steps of preflow-push:
  - Capacity constraint : For all $u, v \in V$, $f(u,v) \leq c(u,v)$

    (same as before)
  - Skew symmetry : For all $u, v \in V$, $f(u,v) = -f(v,u)$

    (same as before)
  - Flow constraint: For all $v \in V - \{s\}$, $\sum_{u \in V} f(u,v) \geq 0$

    (Relaxed, allows net flow into v to be greater than 0)
- Excess flow into v, $e(v)$ = net flow into v = $\sum_{v \in V} f(u,v)$
- A vertex is called active or overflowing if $e(v) > 0$
- f is called a preflow

An Example Preflow

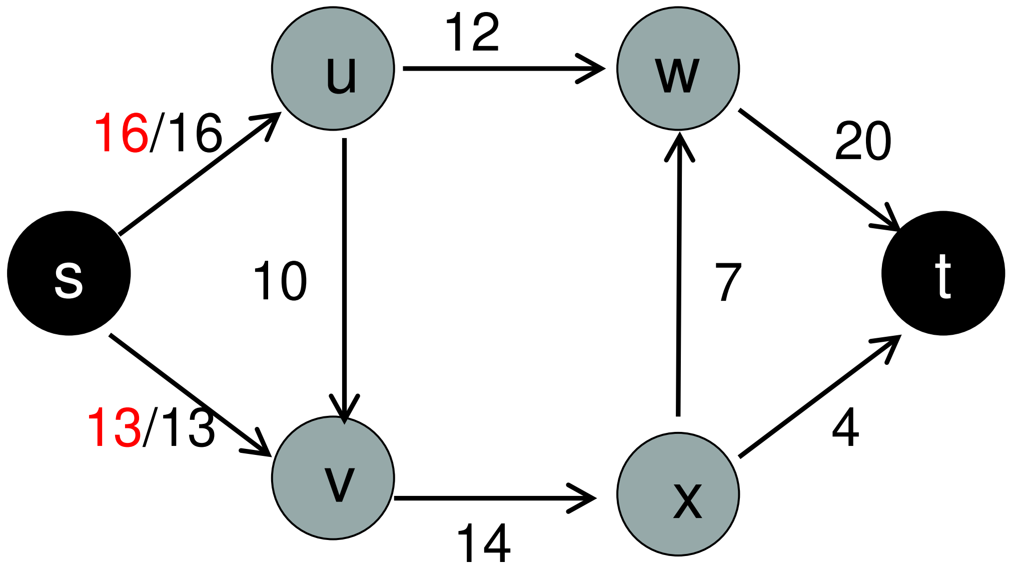


- $e(u) = 2$ (active)
- $e(v) = 4$ (active)
- $e(w) = 2$ (active)
- $e(x) = 0$

Basic Idea

- Think of the vertices at different heights
  - Initially s is at height |V| and all others at height 0
- Think that each vertex has an arbitrarily large temporary storage
- Flow is pushed only downhill, from a vertex with higher height to a vertex with lower height
- Start the algorithm by pushing as much flow as possible from s to all its outgoing edges (i.e., push up to capacity of each edge from s)
  - Initial preflow
- The flow pushed first gets stored in the storage of the vertices at the other end

Initial Preflow



- e(u) = 16 (active)
- e(v) = 13 (active)
- e(w) = 0
- e(x) = 0

- Any other vertex u pushes this flow along each edge whenever possible (if the vertex v at the other end of the edge is at a lower height, i.e, is downhill, and the edge (u,v) is not saturated)
  - PUSH operation
- What if no such vertex v is found?
  - All vertices at the other end of outgoing edges have height $\geq$ this node's height
  - In this case, increase vertex u's height by 1 + minimum height of any vertex at other end of an unsaturated edge
  - RELABEL operation

- Continue until flow cannot be pushed forward anymore
  - All edges across the minimum cut get saturated
- But now you may have vertices with excess flow left in them
- Push this flow back towards s
  - RELABEL to heights greater than |V|
  - Eventually all excess flows go out through s (whose height always stays at |V|)
- The final flow satisfies the flow conservation constraint at each vertex
- So two types of operation, PUSH and RELABEL
  - This is why preflow-push method is also called the push-relabel method

The Height Function

- The same notion of residual capacity c_f and residual graph G_f as before is also used here
- Given a preflow f, a function $h: V \to N$ is a height function if it satisfies the following properties:
  - $h(s) = |V|$
  - $h(t) = 0$
  - $h(u) \leq h(v) + 1$ for any residual edge $(u,v) \in E_f$

- It is usually called the distance function, as it gives a lower bound on the distance from u to t in G_f
  - The text uses the term height to relate to downhill-uphill analogy, so let us use it also
- Note that the definition implies that given any preflow f, for any two vertices u, v, if $h(u) > h(v) + 1$, then (u,v) is not an edge in the residual graph G_f

PUSH Operation

- PUSH(u,v)

    Precondition:

    $e(u) > 0$ (i.e., u is active)

    $c_f(u,v) > 0$

    $h(u) = h(v) + 1$

    Action:

    Let $d_f(u,v) = \min(e(u), c_f(u,v))$

    Push $d_f(u,v)$ amount of flow from u to v

- PUSH is saturating if $c_f(u,v) = 0$ after the PUSH, otherwise non-saturating

PUSH(u, v)

$\triangleright$ **Applies when**: u is overflowing, $c_f(u, v) > 0$, and $h[u] = h[v] + 1$.
$\triangleright$ **Action:** Push $d_f(u, v) = \min(e[u], c_f(u, v))$ units of flow from u to v.
$d_f(u, v) \leftarrow \min(e[u], c_f(u, v))$
$f[u, v] \leftarrow f[u, v] + d_f(u, v)$
$f[v, u] \leftarrow -f[u, v]$
$e[u] \leftarrow e[u] - d_f(u, v)$
$e[v] \leftarrow e[v] + d_f(u, v)$

RELABEL Operation

- RELABEL(u)

  Precondition:

  $e(u) > 0$ (i.e., u is active)

  $h(u) \leq h(v)$ for all edges $(u,v) \in E_f$

  Action:

  $h(u) = 1 + \min\{h(v) \mid (u,v) \in E_f\}$

- Note that h(u) never decreases for any vertex u

RELABEL(u)

▷ **Applies when:** u is overflowing and for all $v \in V$ such that $(u, v) \in E_f$,
we have $h[u] \leq h[v]$.

▷ **Action:** Increase the height of u.

$h[u] \leftarrow 1 + \min \{h[v] : (u, v) \in E_f\}$

An Important Property

For any active vertex u, either a PUSH or a RELABEL operation must be applicable

- Why?
  - If PUSH operation is not applicable, then for all residual edges $(u,v) \in E_f$, $h(u) < h(v) + 1$
    - Note that $h(u)$ cannot be $>$ than $h(v) + 1$ by defn. of h
  - So $h(u) \leq h(v)$
  - But then a RELABEL operation is applicable to u

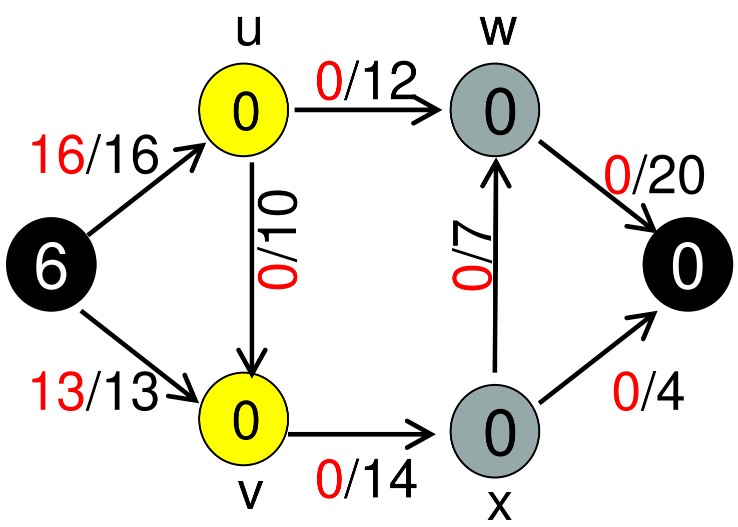
Generic Preflow-Push Algorithm

```
INITIALIZE-PREFLOW(G, s)
for each vertex u ∈ V[G]
      do h[u] ← 0
         e[u] ← 0
for each edge (u, v) ∈ E[G]
      do f[u, v] ← 0
         f[v, u] ← 0
h[s] ← |V[G]|
for each vertex u ∈ Adj[s]
      do f[s, u] ← c(s, u)
         f[u, s] ← −c(s, u)
         e[u] ← c(s, u)
         e[s] ← e[s] − c(s, u)
```

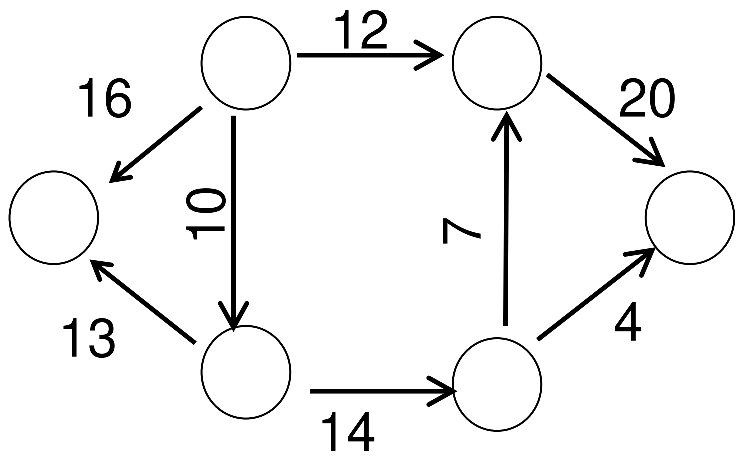
```
GENERIC-PUSH-RELABEL(G)
INITIALIZE-PREFLOW(G, s)
while there exists an applicable push or relabel operation
      do select an applicable push or relabel operation and perform it
```

Example

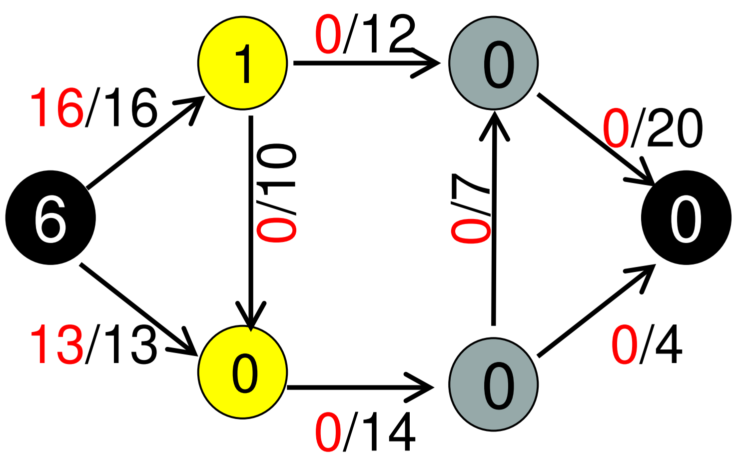
Initial Preflow

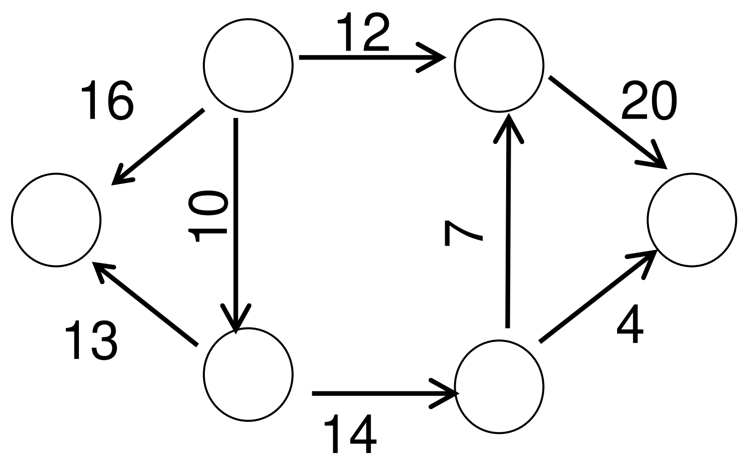


Residual Graph

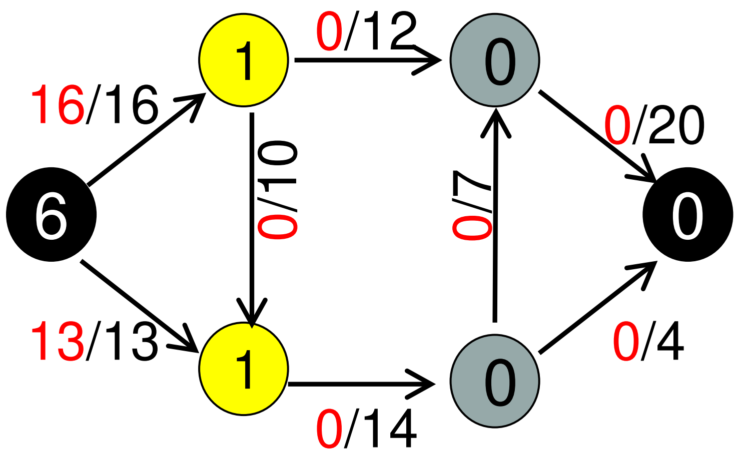


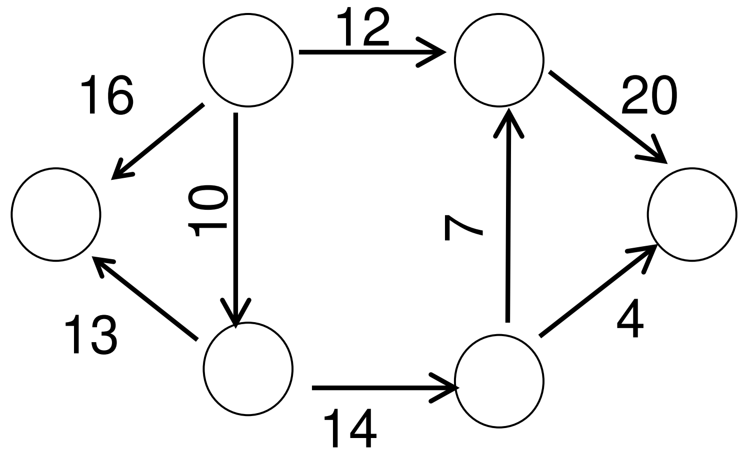
RELABEL(u)



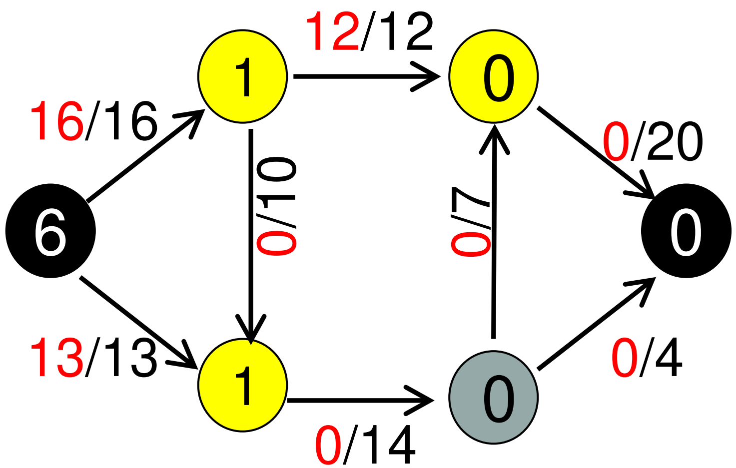


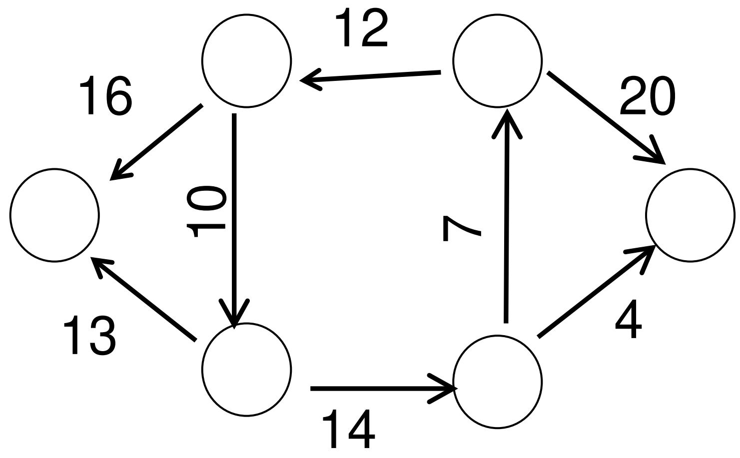
RELABEL(v)



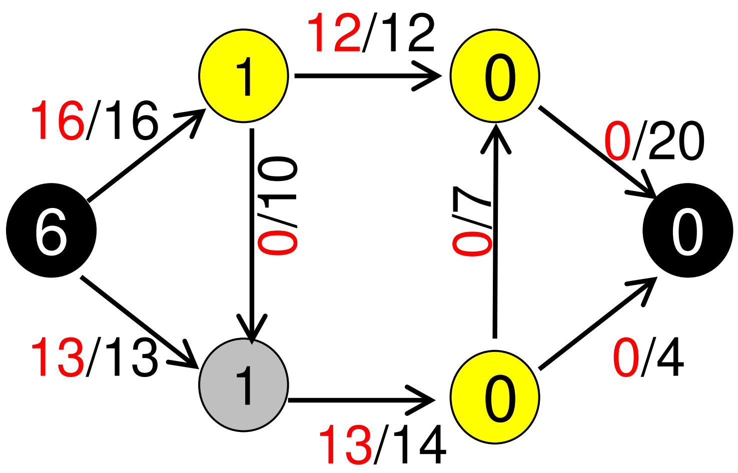


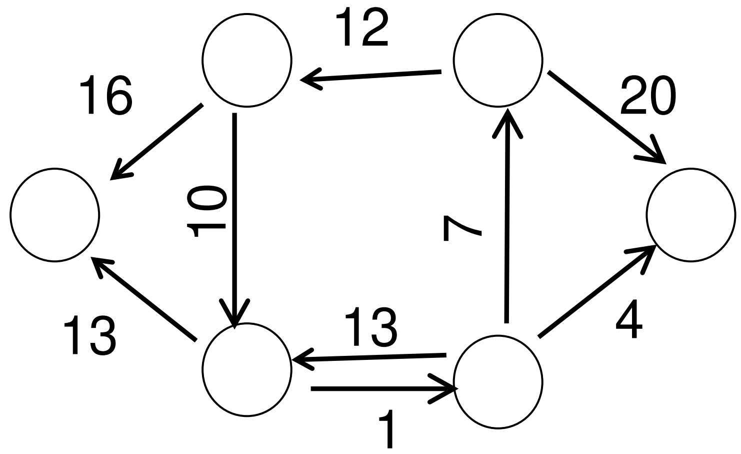
PUSH(u,w)



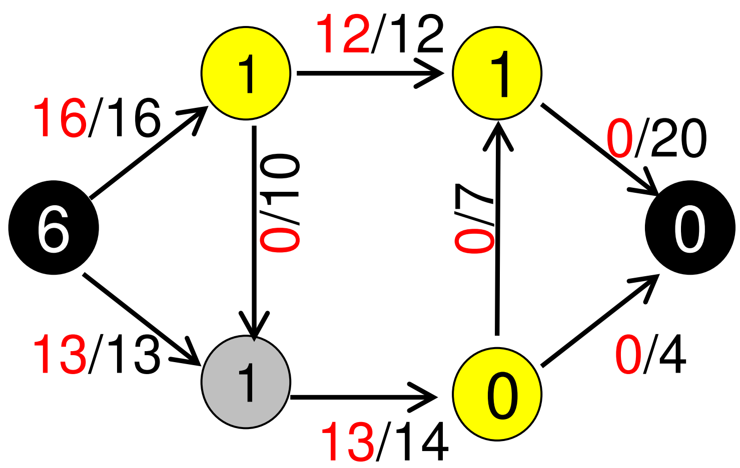


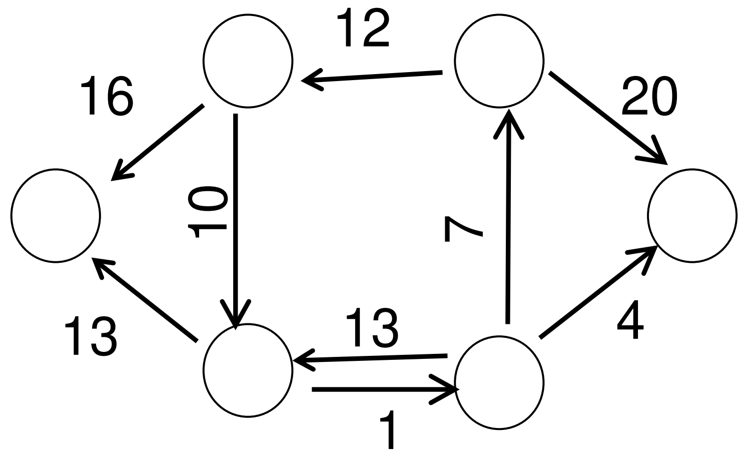
PUSH(v,x)



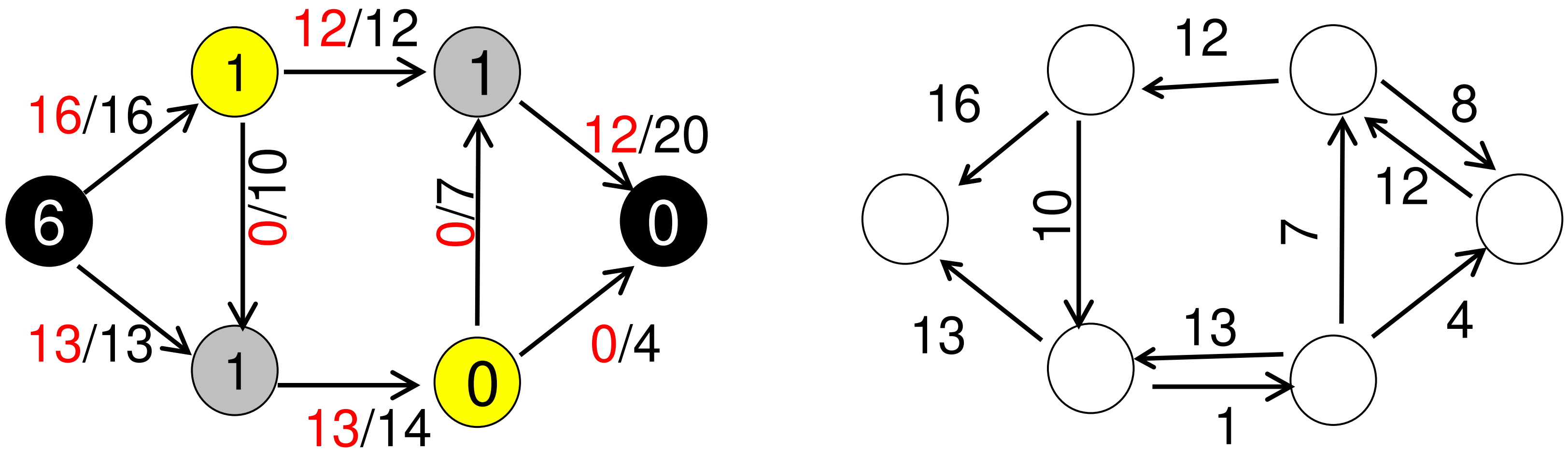


RELABEL(w)

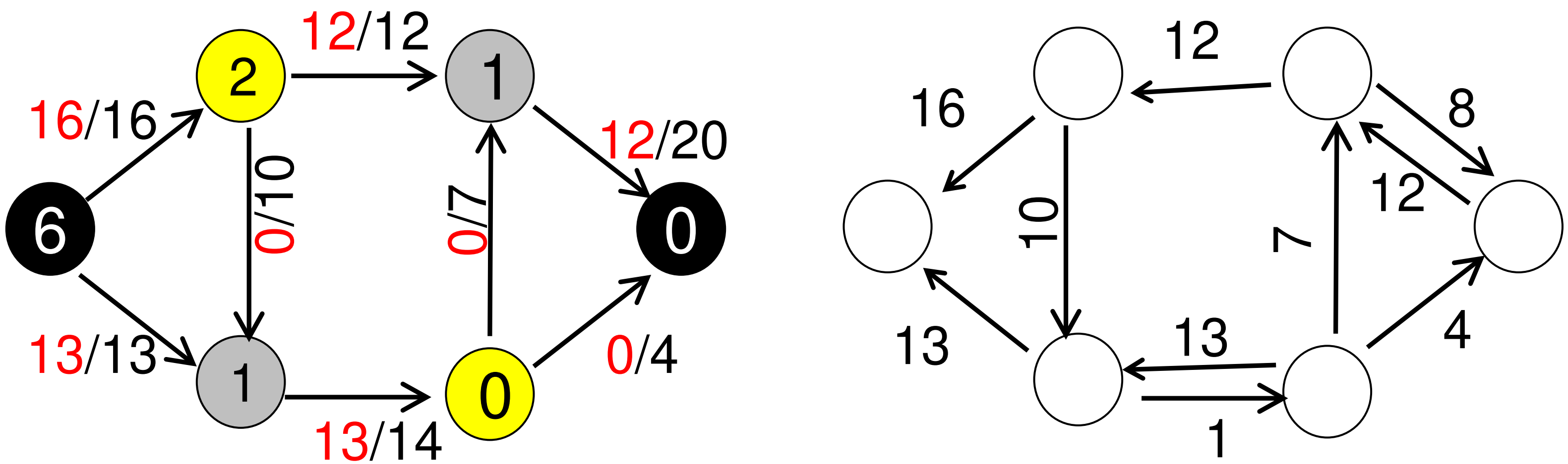




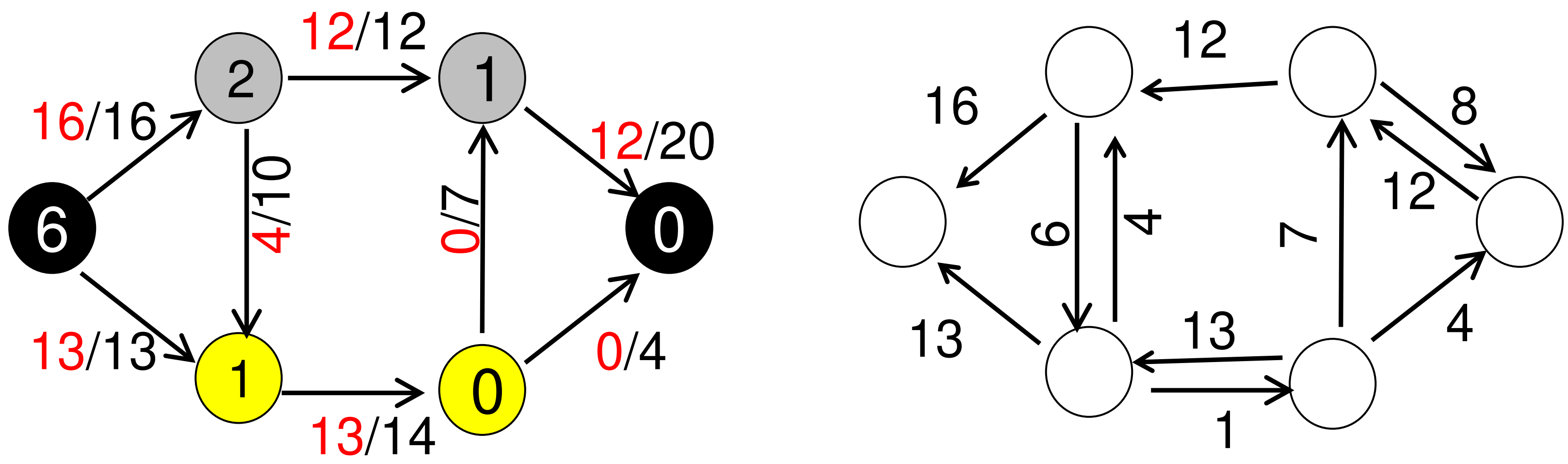
PUSH(w, t)



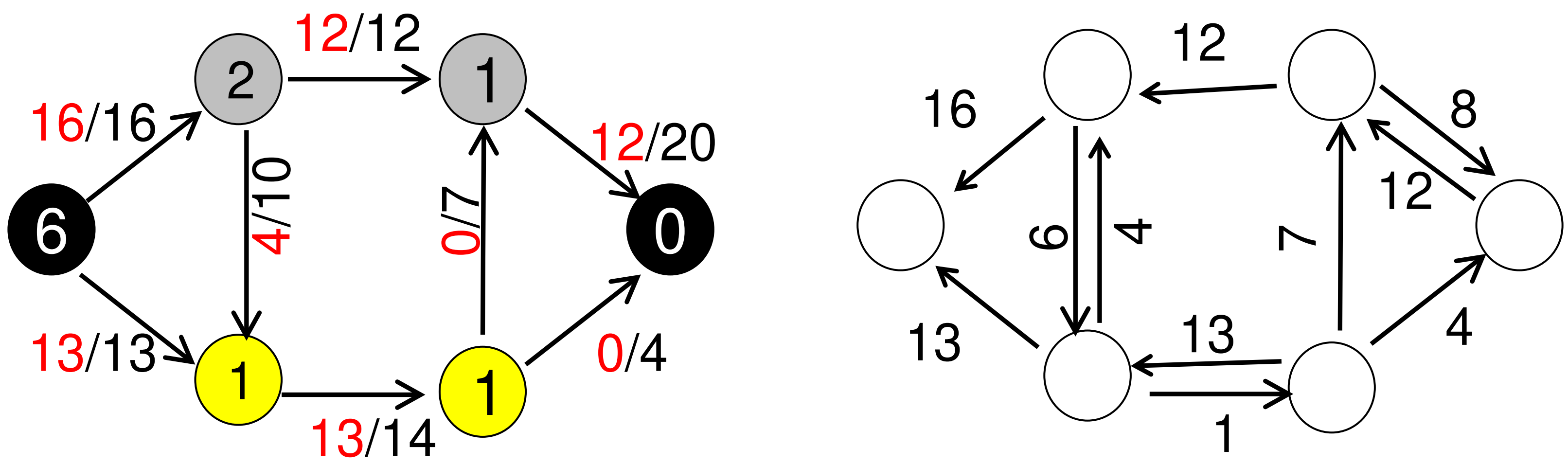
RELABEL(u)



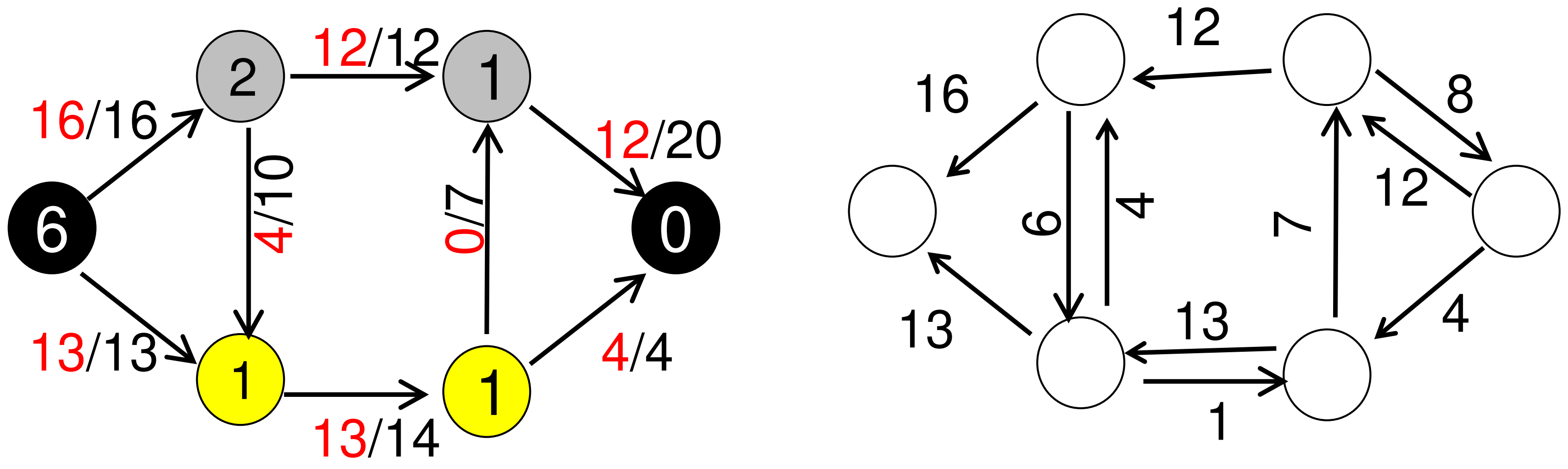
PUSH(u,v)



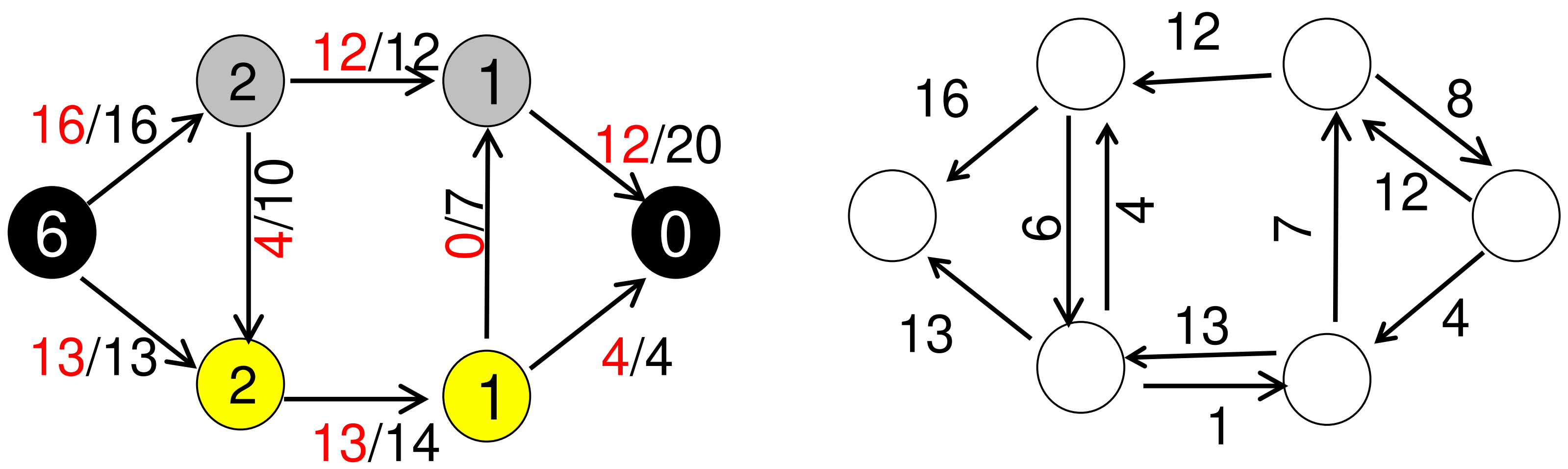
RELABEL(x)



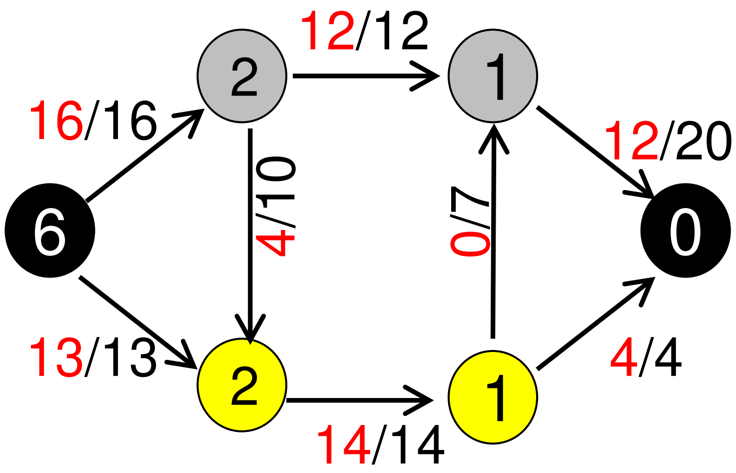
PUSH(x, t)

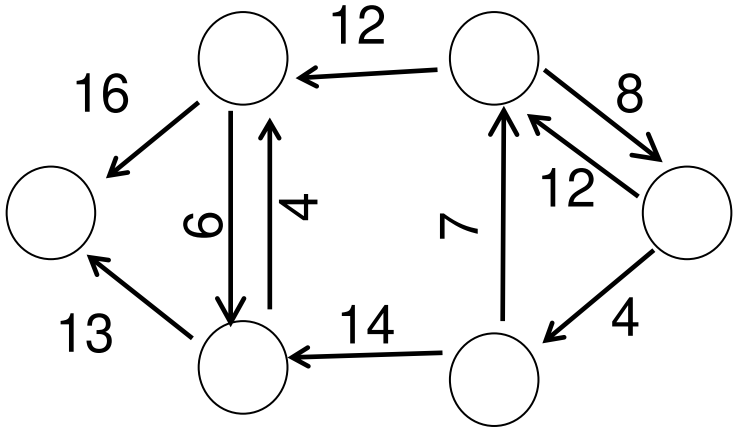


RELABEL(v)

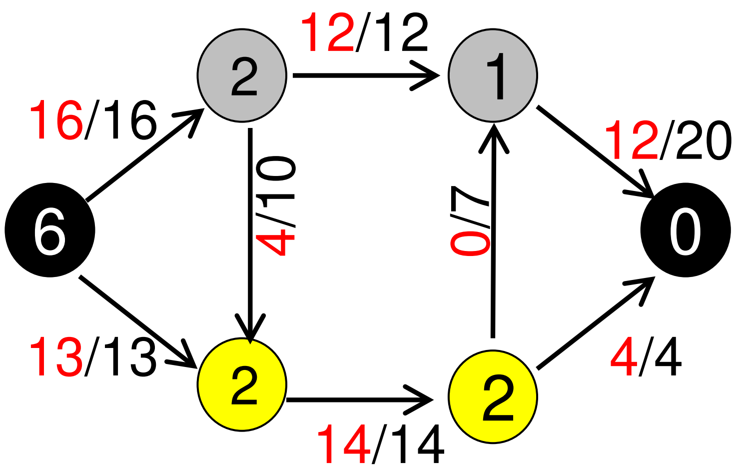


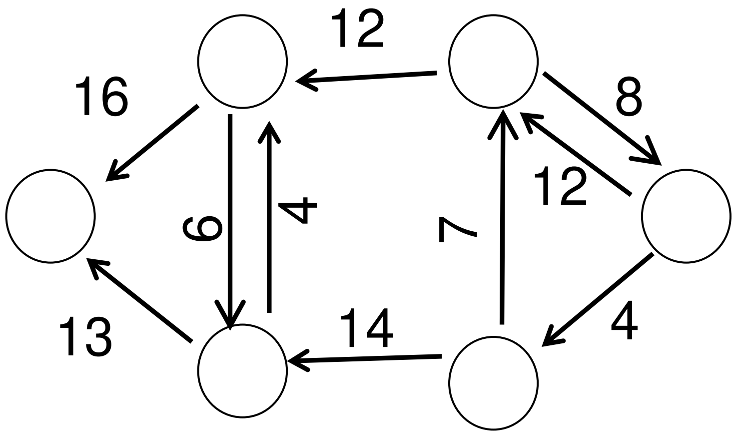
PUSH(v,x)



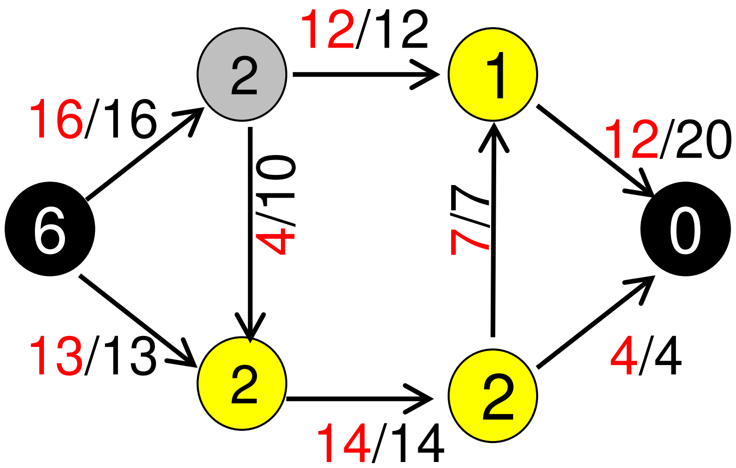


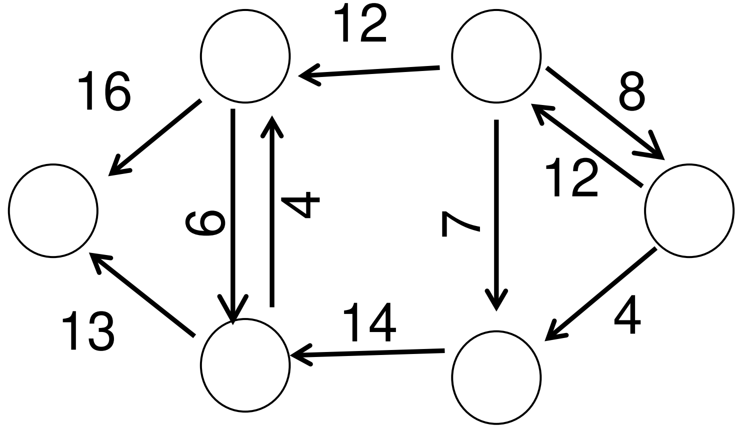
RELABEL(x)



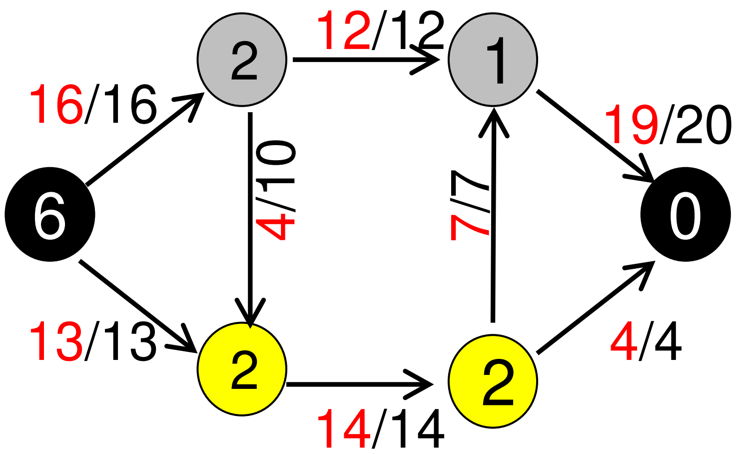


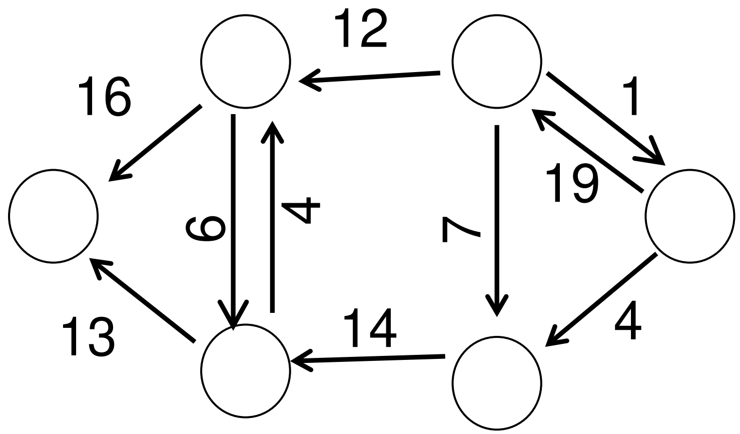
PUSH(x,w)



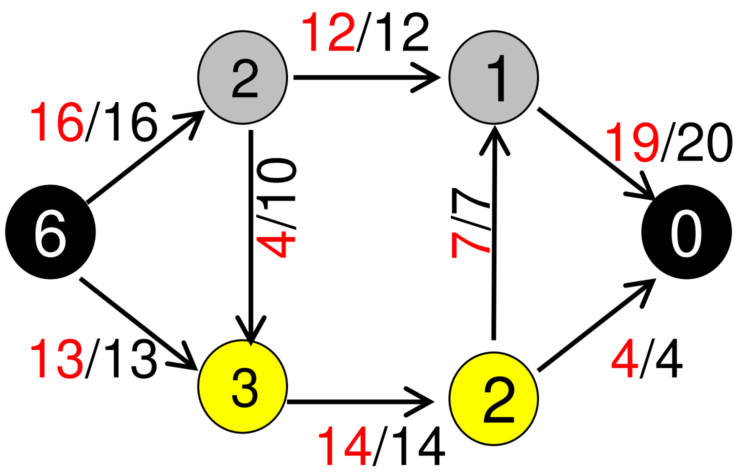


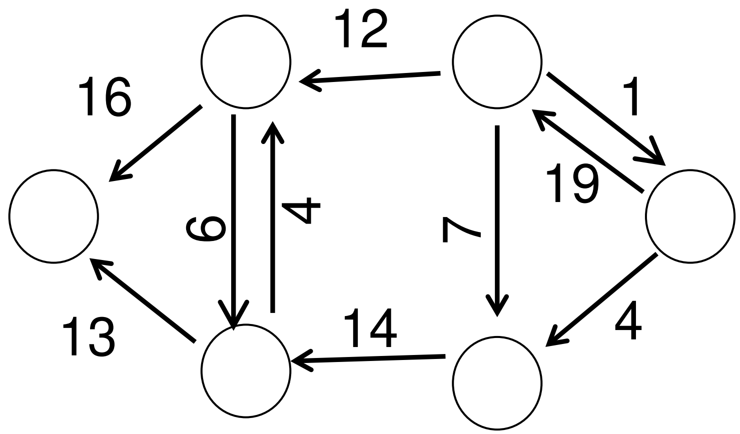
PUSH(w,t)



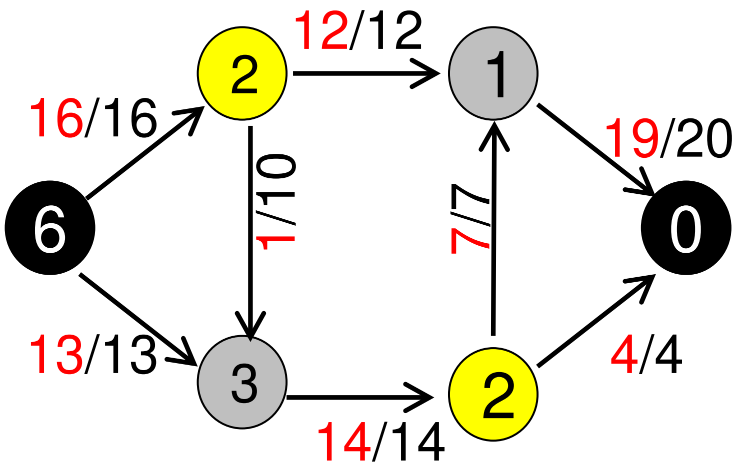


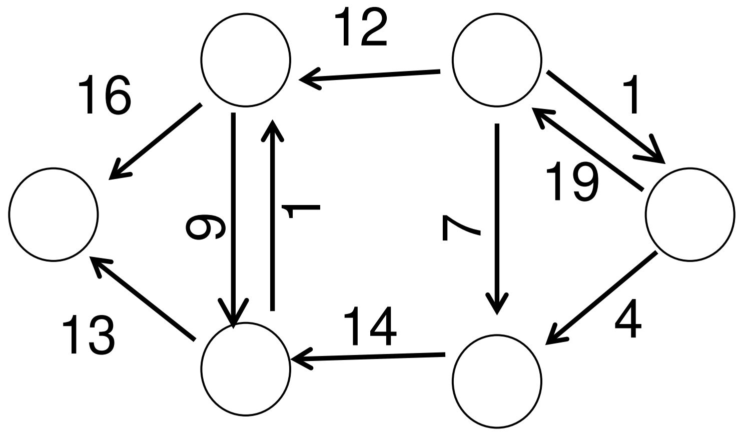
RELABEL(v)



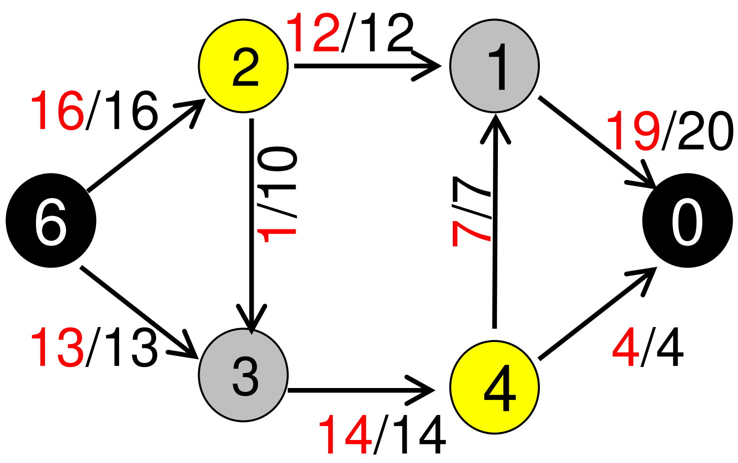


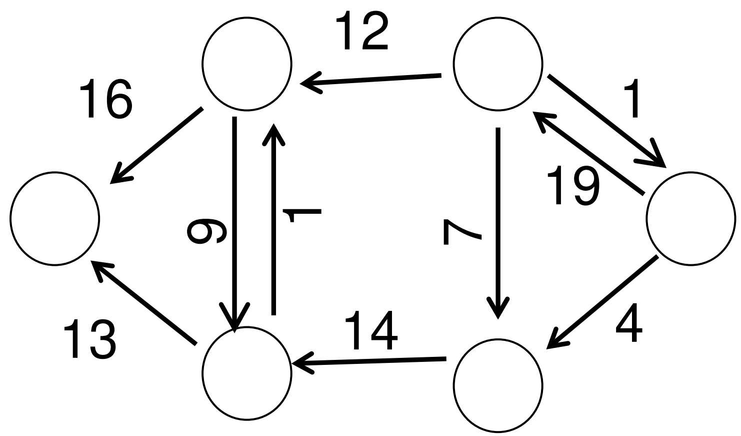
PUSH(v,u)



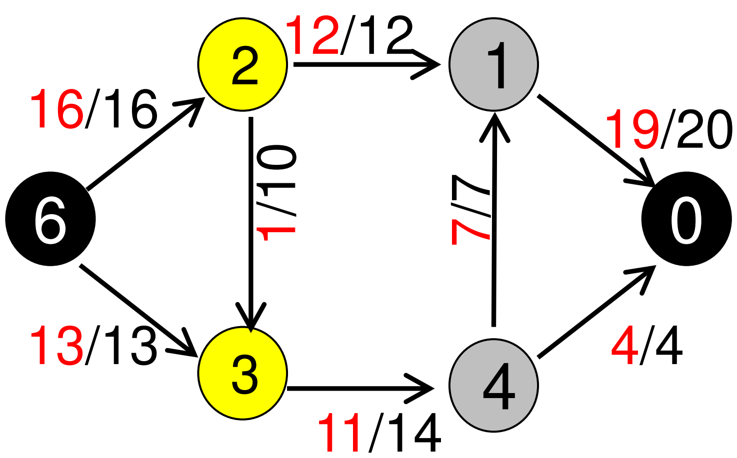


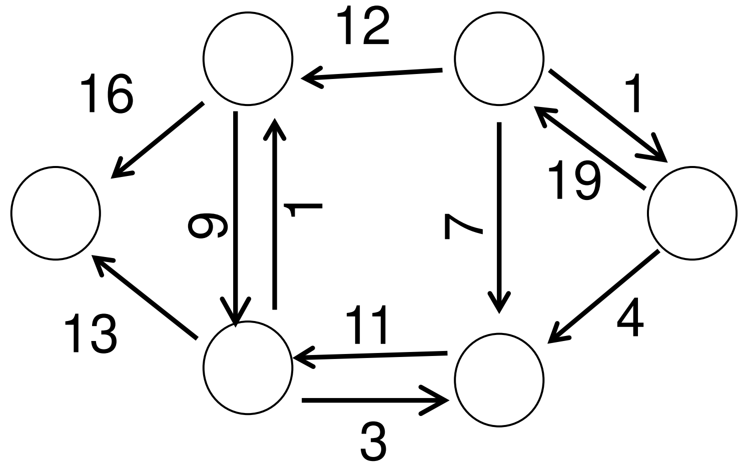
RELABEL(x)



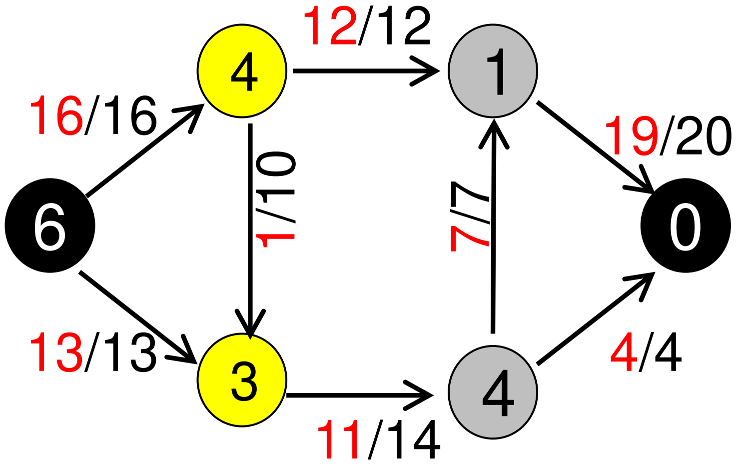


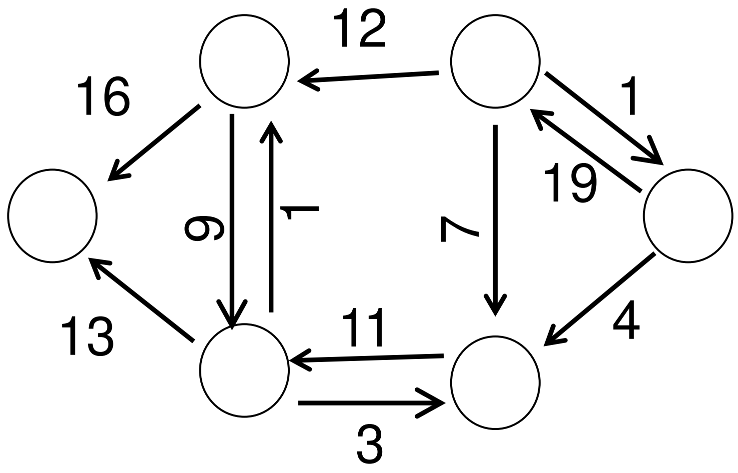
PUSH(x,v)



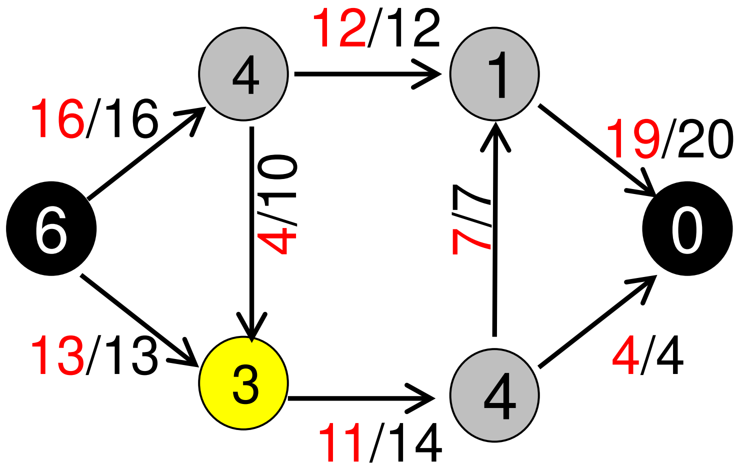


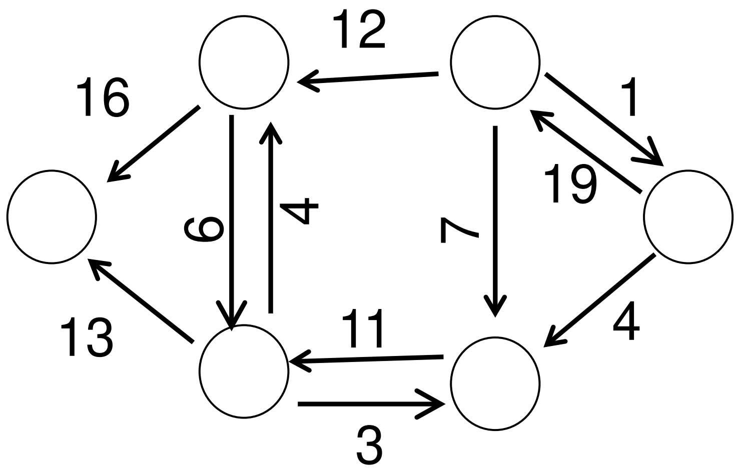
RELABEL(u)



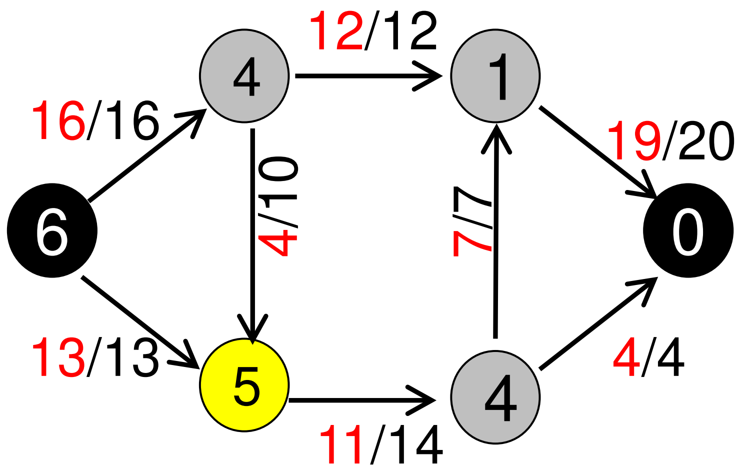


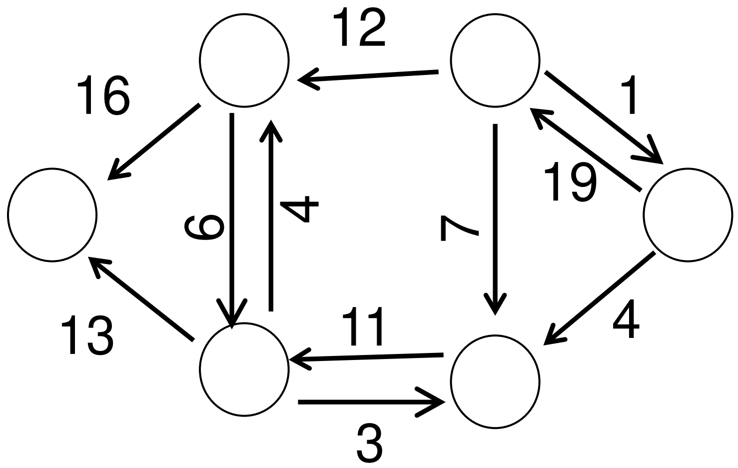
PUSH(u,v)



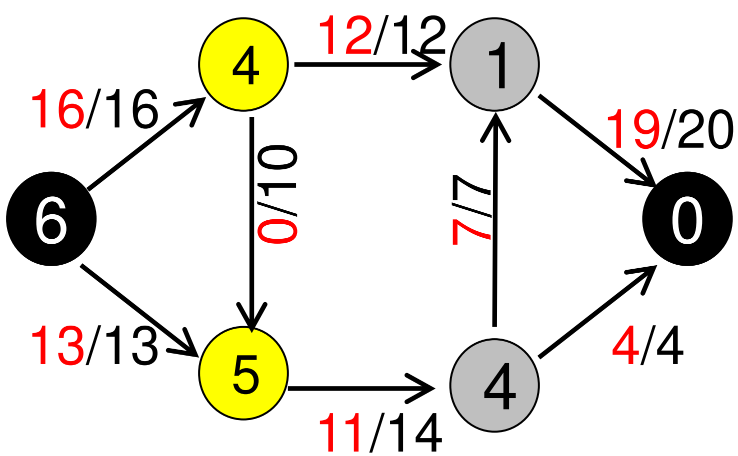


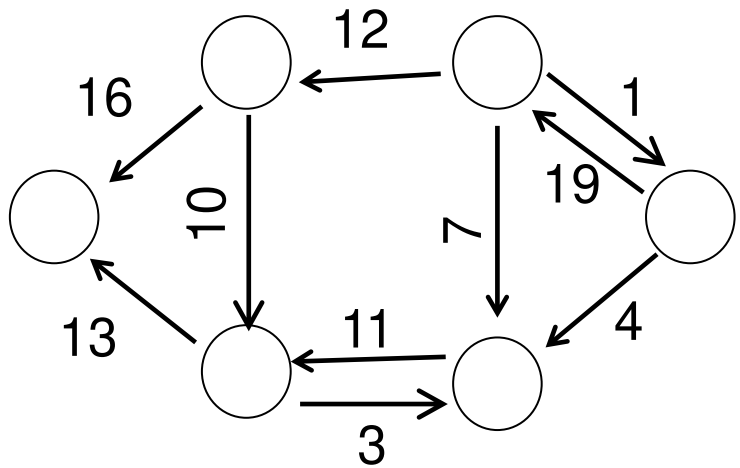
RELABEL(v)



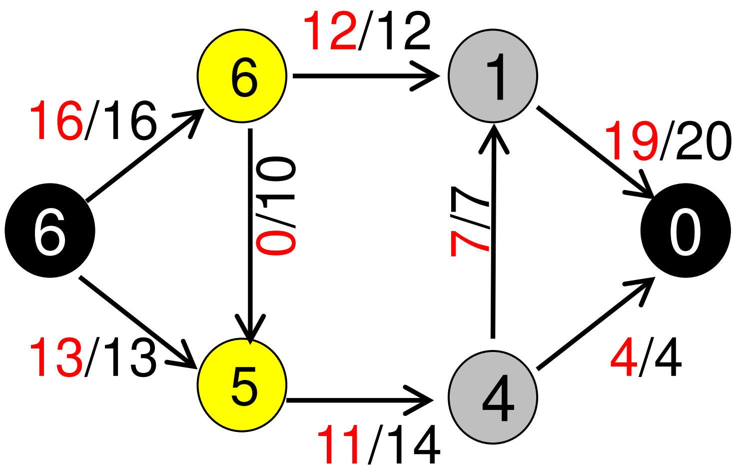


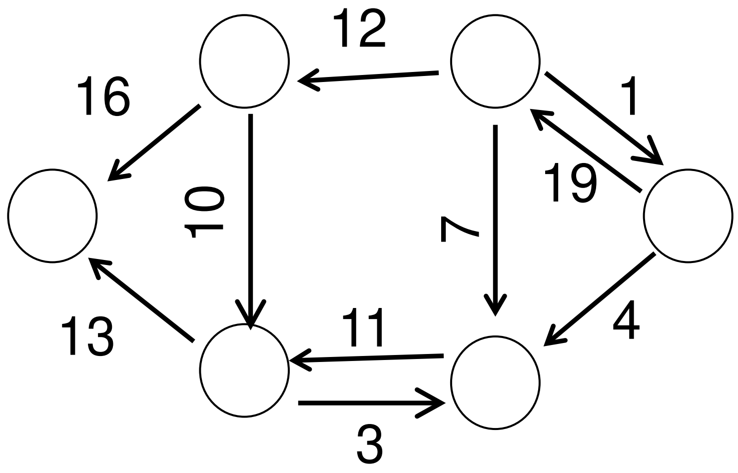
PUSH(v,u)



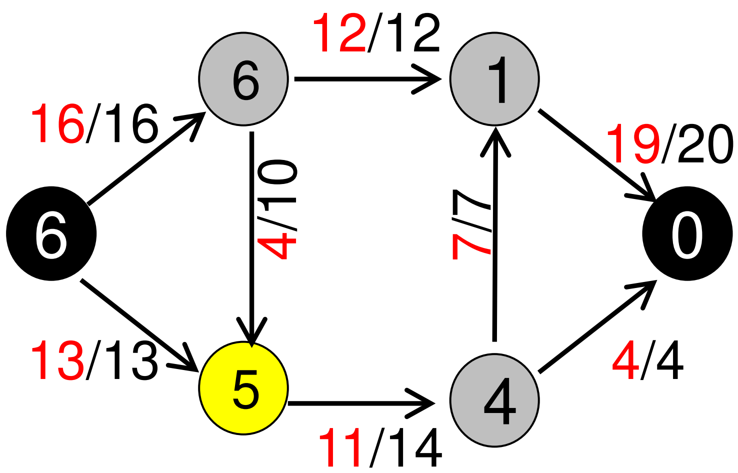


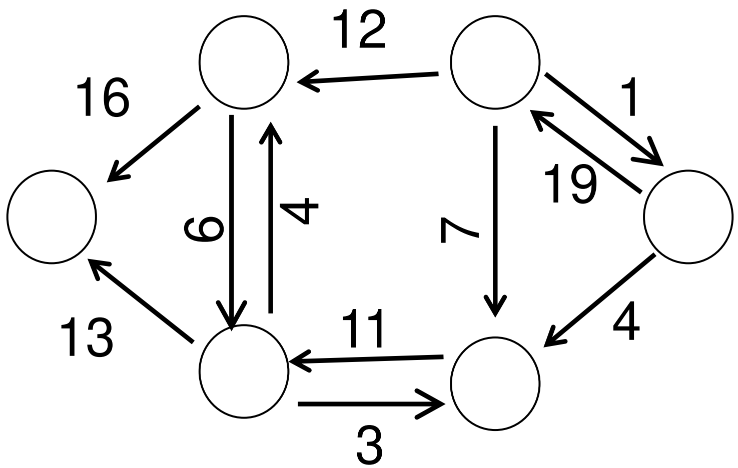
RELABEL(u)



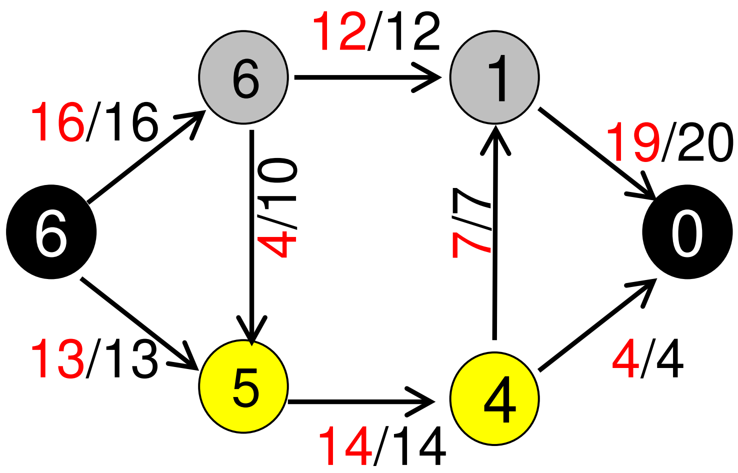


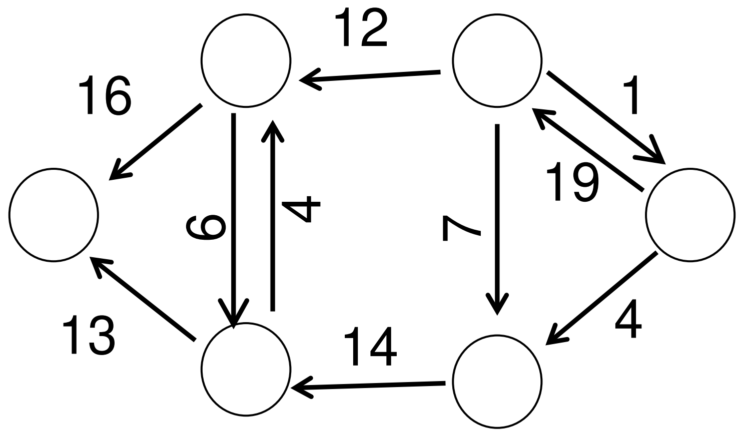
PUSH(u,v)



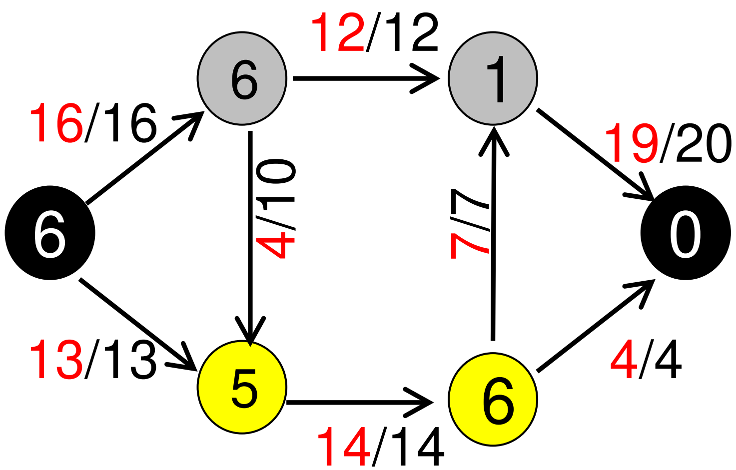


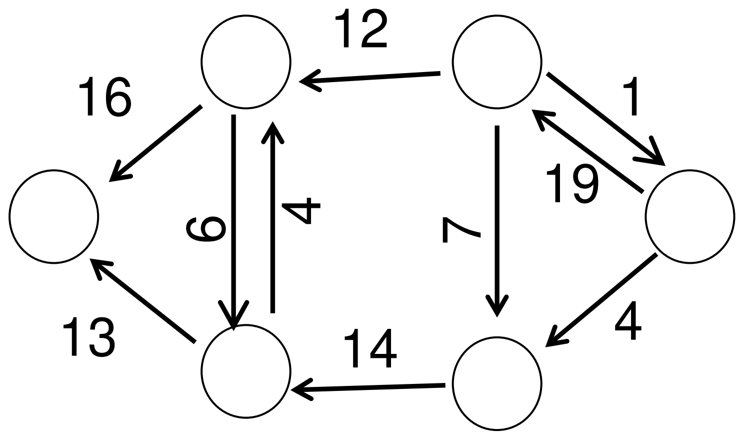
PUSH(u,x)



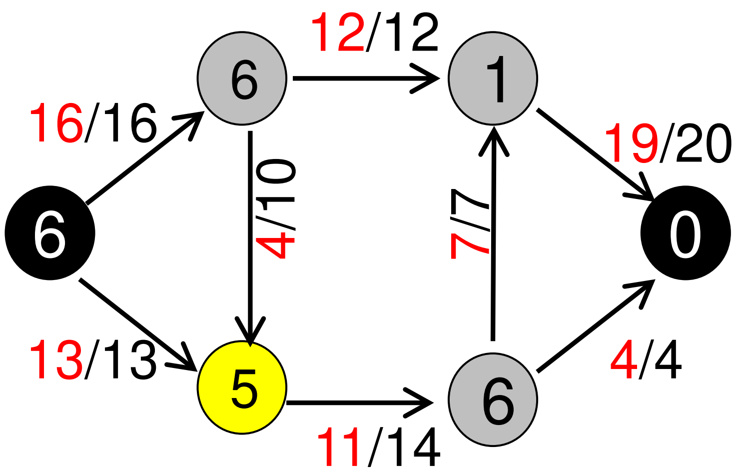


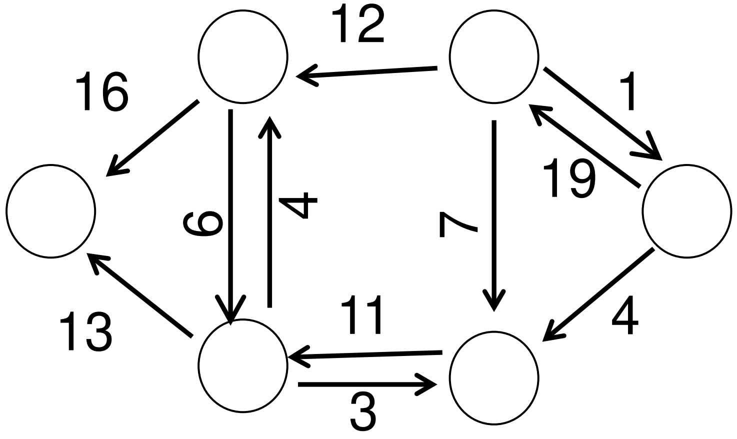
RELABEL(x)



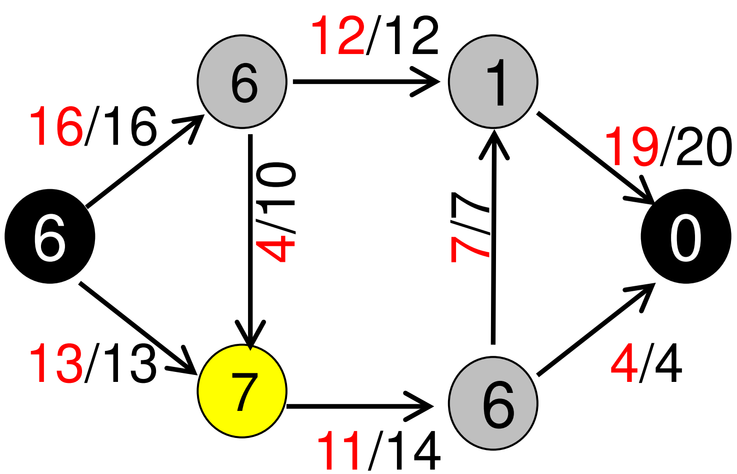


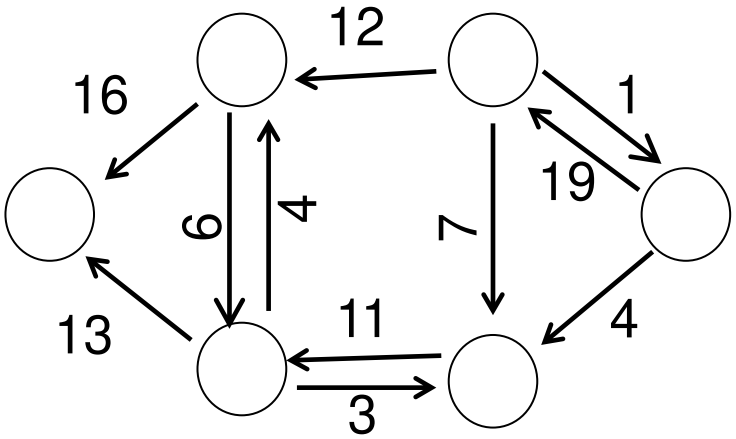
PUSH(x,v)



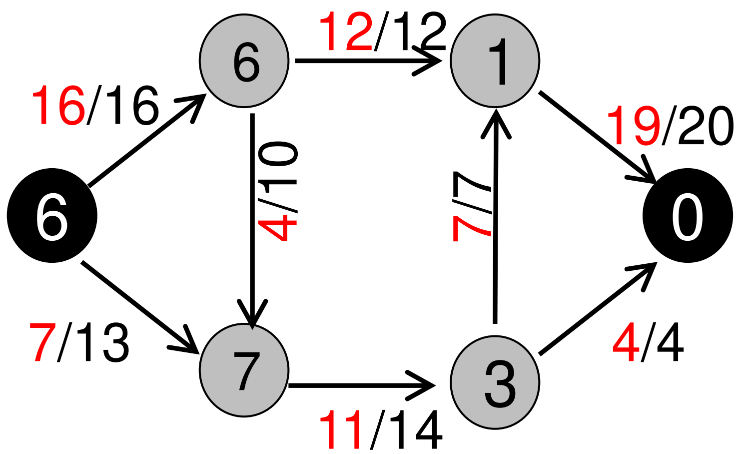


RELABEL(v)





PUSH(v,s)



No active node, so stop

Maximum flow $|f| = 23$

Proof of Correctness (Outline)

- Claim 1: Vertex heights never decrease
  - PUSH does not change h, and RELABEL only increases it
- Claim 2: PUSH(u,v) and RELABEL(u) maintain the properties of the height function
  - PUSH(u,v) pushes flow along $(u,v) \in E_f$, so there may be two possibilities:
    - It may add the edge (v,u) to E_f. Since PUSH(u,v) occurred, so $h(u) = h(v) + 1$ before the push. PUSH does not change h. So $h(v) = h(u) - 1 < h(u) + 1$ after the push, which satisfies the height function property for the edge (v,u)
    - It may remove the edge (u,v) from E_f. Then the constraint does not apply to (u,v) anyway (as height function properties apply only for edges in E_f)

- RELABEL(u) increases h(u)
  - Outgoing edges from u in G_f: Just before relabel, $h(u) \leq h(v)$ for any edge $(u,v) \in E_f$. Relabel increases h(u) to 1 + minimum of the h(v)'s. So $h(u) \leq h(v) + 1$ for any edge $(u,v) \in E_f$. This satisfies the height function property.
  - Incoming edges to u in G_f: For any edge $(w,u) \in E_f$, just before RELABEL, $h(w) \leq h(u) + 1$ (as the height function was satisfied before the operation). So just after RELABEL, $h(w) < h(u) + 1$ trivially as h(u) is increased.

- Claim 3: For a preflow f, there is no path from s to t in the residual graph G_f
  - Can show by contradiction
    - Assume that such a path p exists. By the property of the height function, for any edge $(u,v) \in E_f$, $h(u) \leq h(v) + 1$. Applying this to successive vertices of the path p, it is easy to show that $h(s) \leq h(t) + k$, where k is the length of the path. But that means h(s) cannot be $|V|$, as $h(t) = 0$ and $k < |V|$. This is a contradiction.

- Claim 4: PUSH operations maintains the properties of a preflow
  - Since PUSH increases flow from u to v by $d_f(u,v)$ = min(e(u), $c_f(u,v)$) amount, it cannot make e(u) negative or exceed the capacity c(u,v). So the preflow f after the PUSH satisfies the capacity constraint and the flow constraint. It obviously satisfies the skew symmetry constraint (see pseudocode). So if f is a preflow before the PUSH, it remains a preflow after the PUSH

Theorem: If the algorithm terminates, the preflow f at the end is a maximum flow.

Proof Outline:

- Initial f is a preflow.
- RELABEL operations do not affect flow, so a preflow remains a preflow
- PUSH operations also maintain preflows (Claim 4)
- Termination means for any vertx in V – {s,t}, PUSH and RELABEL are not applicable, which implies all vertices in V – {s,t} must have excess 0. So it is a flow, and it will not change (as no more PUSH and RELABEL can be done)
- We know that there is no path from s to t in G_f (Claim 3)
- So there is no augmenting path in the residual graph, so by max-flow min-cut theorem, f is a maximum flow.

- Are we done with correctness proof?

- No. We have proved "If" it terminates, f is a maximum flow
- We have not proved that it "does" terminate
  - What if there is always one or more vertices with excess > 0, and an infinite sequence of PUSH and RELABEL operations occur?
- So we have to prove that the algorithm terminates
  - We can prove termination by showing that the number of PUSH and the number of RELABEL operations are bounded

- We will omit this proof, will just note that the following can be proved:
  - At any time t during the execution of the algorithm, $h(u) \leq 2|V| - 1$
  - Then, the number of RELABEL operations is bounded by $(2|V| - 1)(|V| - 2) < 2|V|^2$
  - Number of saturating pushes is $< 2|V||E|$
  - Number of nonsaturating pushes is $< 4|V|^2(|V| + |E|)$
  - Therefore time complexity $= O(|V|^2E)$
    - Can implement each PUSH and RELABEL in O(1) time

- Note that the algorithm we presented is "generic" in the sense that it can apply PUSH and RELABELs in any order
- There are different implementations that apply these operations in different specific orders to get better complexity
  - Relabel-to-front
  - FIFO
  - Highest-label
  - …..