Some NP-Complete Problems

SAT, CNFSAT

Theorem: 3-CNFSAT is NP-Complete.

CNFSAT $\leq$ 3-CNFSAT

$$\phi \longmapsto \phi'$$

α – a clause in ϕ

$$\alpha = a_1 \vee a_2 \vee a_3 \vee \cdots \vee a_\ell$$

$\ell = 3$, done.

$\ell = 1$, $\alpha = a_1 = a_1 \vee a_1 \vee a_1$

$\ell = 2$, $\alpha = a_1 \vee a_2 = a_1 \vee a_2 \vee a_2$

$l \geq 4. \qquad y_1, y_2, \dots, y_{l-3}$

$$\alpha' = (a_1 \vee a_2 \vee y_1) \wedge (\bar{y}_1 \vee a_3 \vee y_2) \wedge (\bar{y}_2 \vee a_4 \vee y_3) \wedge \dots$$

$$\wedge (\bar{y}_{l-4} \vee a_{l-2} \vee y_{l-3}) \wedge (\bar{y}_{l-3} \vee a_{l-2} \vee a_l)$$

$\alpha = 1 \qquad a_i = 1$

$i = 1, 2 \qquad y_1 = y_2 = \dots = y_{l-3} = 0$

$i = l-1, l \qquad y_1 = y_2 = \dots = y_{l-3} = 1$

$3 \leq i \leq l-2 \qquad y_1 = y_2 = \dots = y_{i-2} = 1$

$y_{i-1} = \dots = y_{l-3} = 0$

$\alpha = 0 \qquad \alpha' = (y_1) \wedge (\bar{y}_1 \vee y_2) \wedge (\bar{y}_2 \vee y_3) \wedge \dots$

$$\wedge (\bar{y}_{l-4} \vee y_{l-3}) \wedge (\bar{y}_{l-3})$$

HAM PATH / CYCLE

Undirected / directed

<u>Theorem</u>: DHAMPATH is NP-Complete.

$$\text{CNFSAT} \leq \text{DHAMPATH}$$

$$\phi \longmapsto G, s, t$$

ϕ is natisfiable $\cancel{\Leftrightarrow}$ G contains an s, t Hamiltonian path

$$\text{CNF} = C_1 \wedge C_2 \wedge C_3 \wedge \cdots \wedge C_k$$

- No clause contains repeated literals
- No clause contains both a variable and its complement

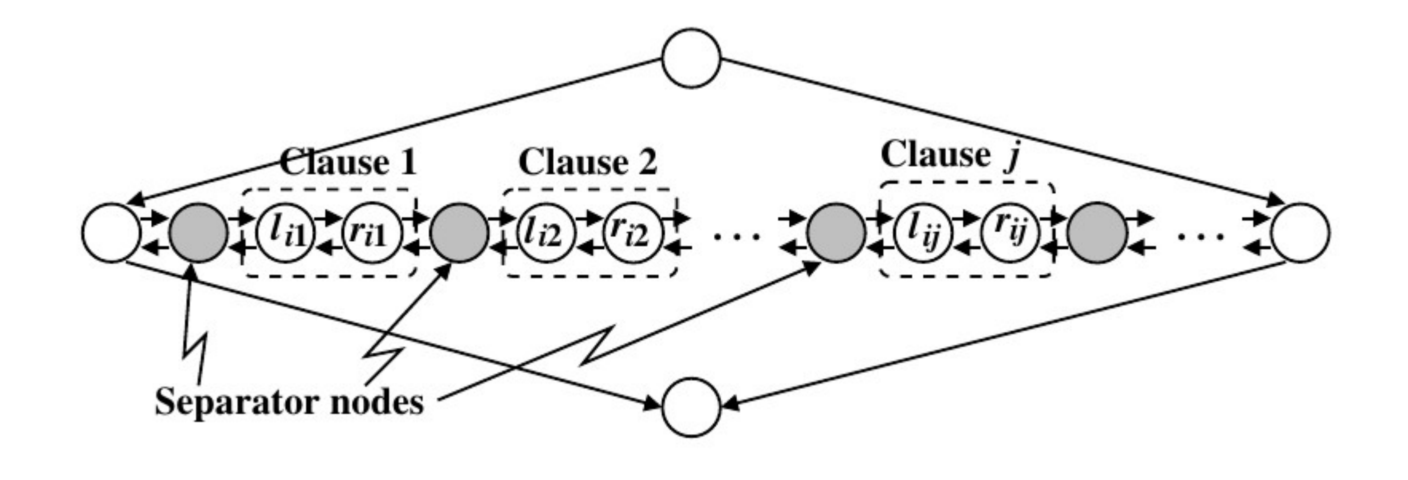
variable — variable gadget

clause — clause gadget

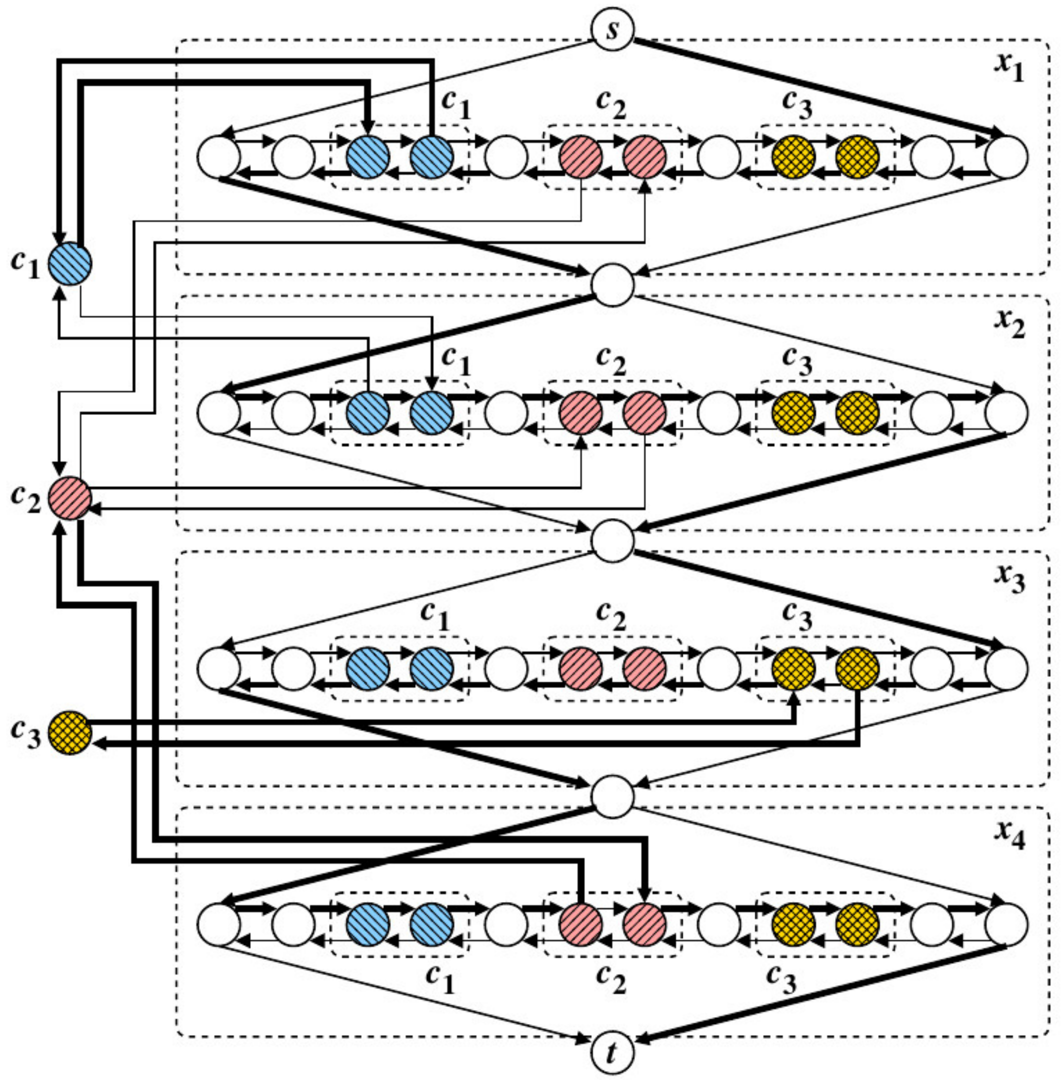
c_i

C_i ○

variable gadget for x_i



Converting $\phi = (\overline{x_1} \vee x_2) \wedge (x_1 \vee \overline{x_2} \vee x_4) \wedge (\overline{x_3})$ to a directed graph for proving the NP-Completeness of the directed Hamiltonian path problem



$x_1 = 0$

$x_2 = 1$

$x_3 = 0$

$x_4 = 1$

If G contains an s,t Ham path, then ϕ is satisfiable

$C_1: \overline{x}_1, x_2$

$C_2: x_4$

$C_3: \overline{x}_3$

VERTEX COVER PROBLEM

$G = (V, E)$ undirected

$U \subseteq V$ is called a vertex cover if every edge $(u, v) \in E$ has at least one endpoint in U.

minimum vertex cover

Given G and a +ve integer ℓ, decide whether G contains a vertex cover U of size $|U| = \ell$.

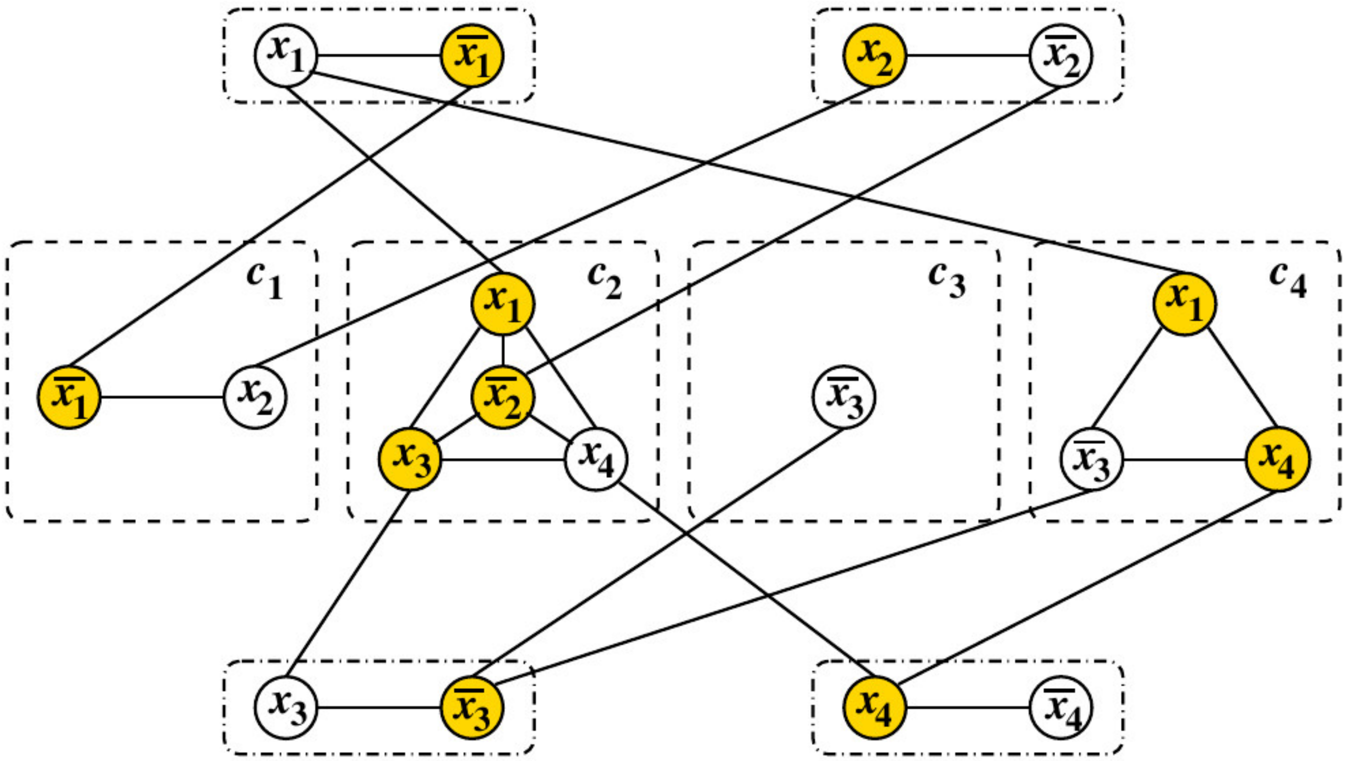
Reduction from CNFSAT

$$\phi \longmapsto G, \ell$$

G contains a vertex cover of size ℓ

$\iff$ ϕ is satisfiable

Converting $\phi = (\overline{x_1} \vee x_2) \wedge (x_1 \vee \overline{x_2} \vee x_3 \vee x_4) \wedge (\overline{x_3}) \wedge (x_1 \vee \overline{x_3} \vee x_4)$ to an undirected graph for proving the NP-Completeness of the vertex cover problem



4

n = # of variables

m = # of clauses 4

t = # of literals in all the clauses

10

ϕ is satisfiable.

$x_1 = 0$
$x_2 = 1$
$x_3 = 0$
$x_4 = 1$

$$\boxed{\ell = n + (t - m)}$$

SAT, CNFSAT

first-generation

Second-generations

Third-generation