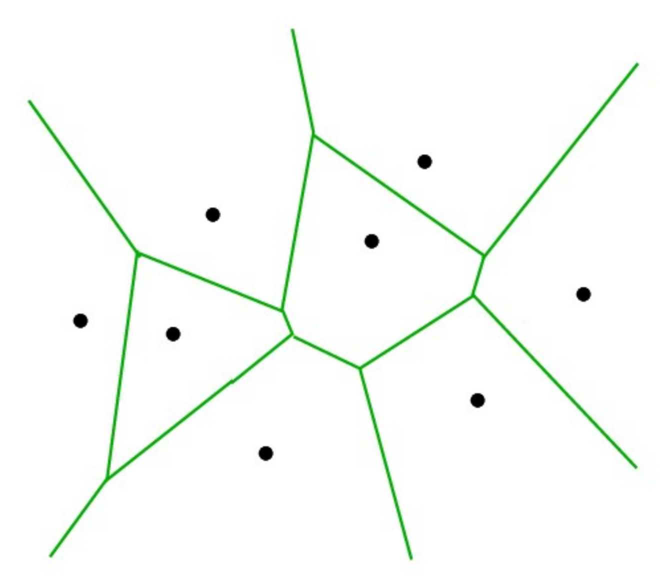
Voronoi Diagrams



Voronoi diagram

Inputs: A set of points in the Euclidean plane

Sites

$S = \{P_1, P_2, \ldots, P_n\}$

$VCell(P_i) = \{Q \mid d(P_i,Q) <= d(P_j,Q)$ for all $j != i\}$

convex region

unbounded region - two semi-infinite lines
bounded - convex polygon

Vor(S)

Naive Algorithm

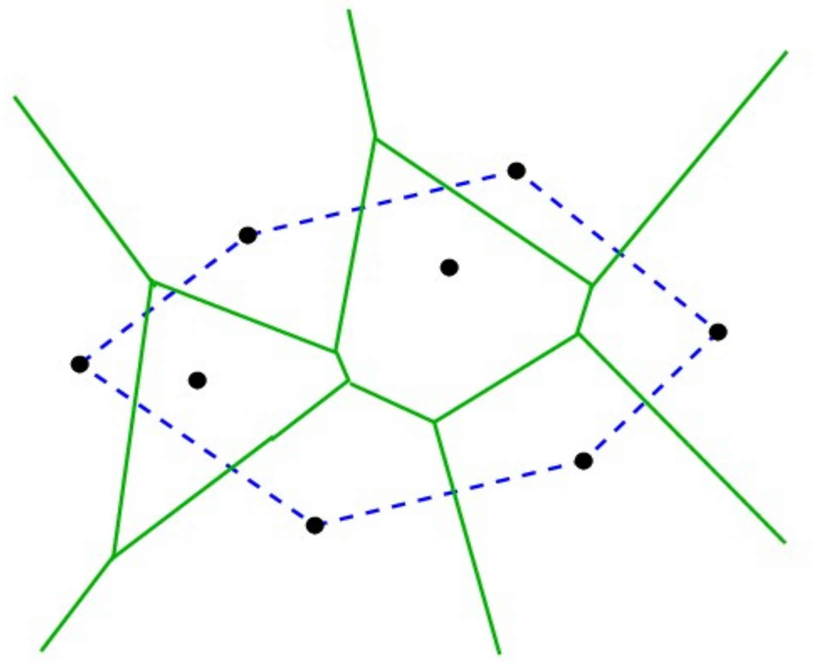
$$H_{ij} = \{ Q \mid d(Q, P_i) \leq d(Q, P_j) \}$$

$$\text{UCell}(P_i) = \bigcap_{j \neq i} H_{ij}$$

$\Omega(n)$ time for each P_i.

$\Omega(n^2)$ time.

Fortune's algorithm



Voronoi diagram and convex hull

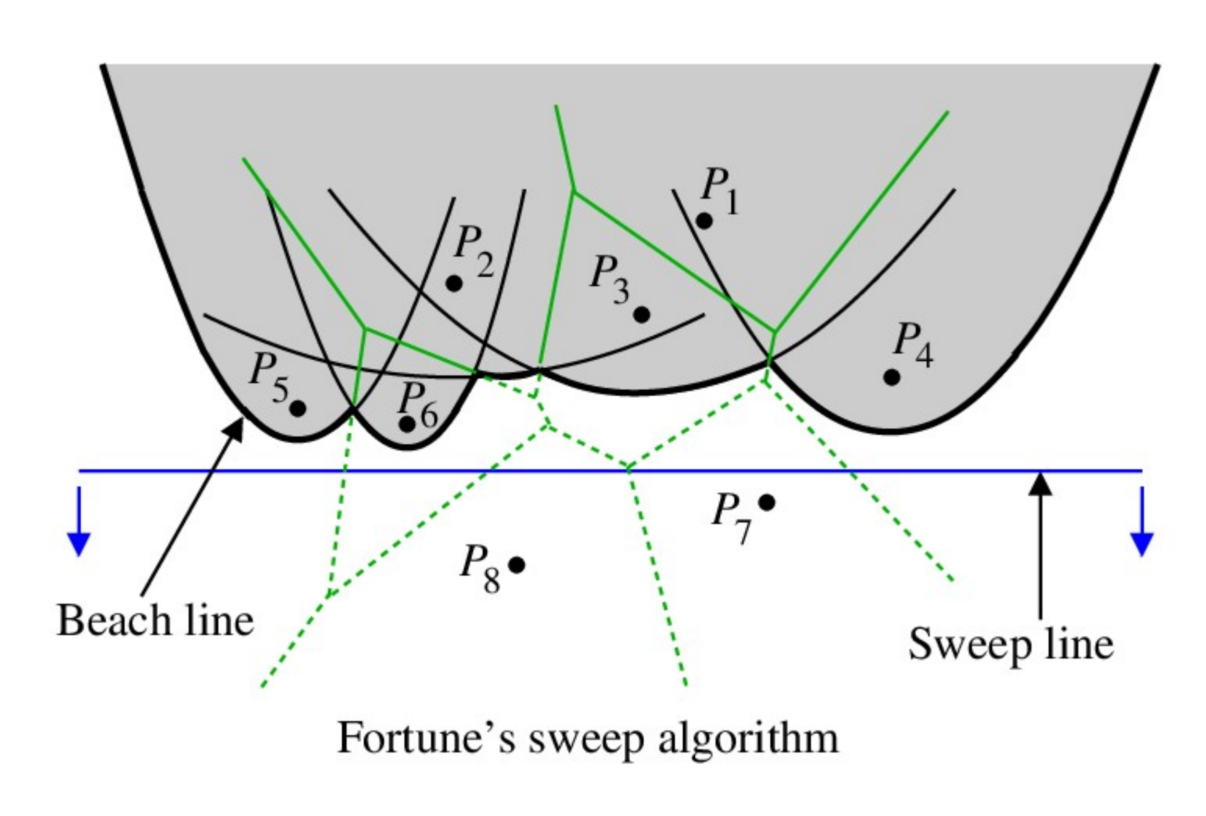
Two Voronoi cells share a semi-infinite line
if and only if
the sites are two consecutive vertices on the convex hull of S.

O(n log n) time algorithm for CH(S)

One ends of the semi-infinite lines are discovered.
Upward semi-infinite line, one end is at +INFINITY
Downward semi-infinite line, one end is at -INFINITY

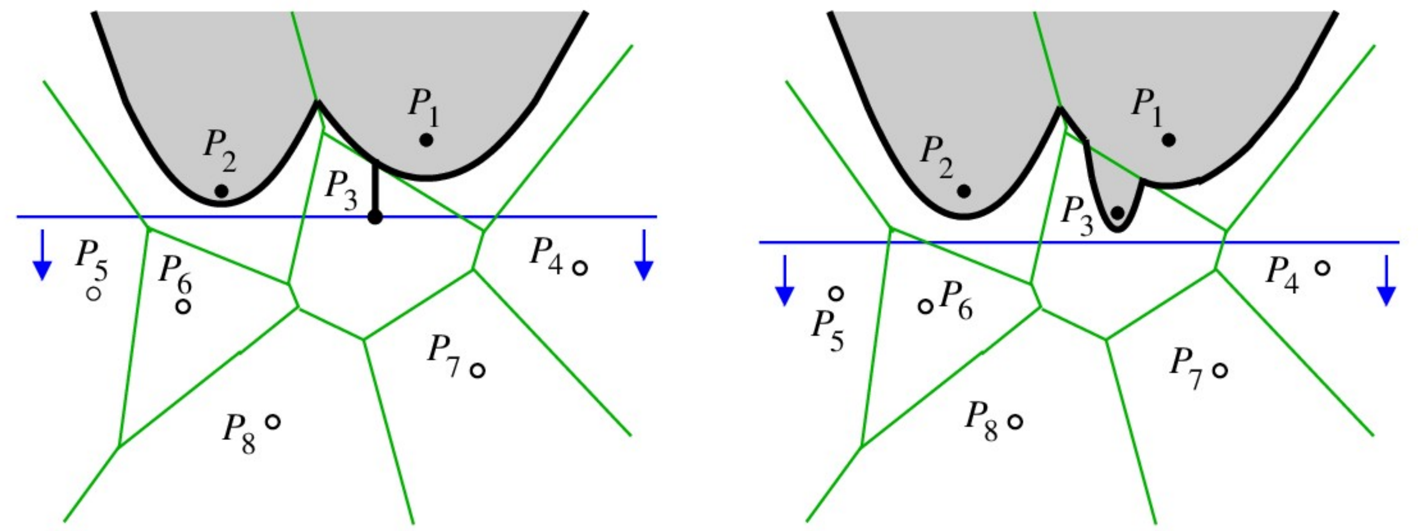
Each segment has three states:
No ends are discovered: Yet to be opened
One end is discovered: Open
Both ends are discovered: Closed

Line sweep algorithm



x-monotone

Site event - Changes the beach line
but does not produce any vertex in Vor(S)
Circle event - All vertices are discovered by these events



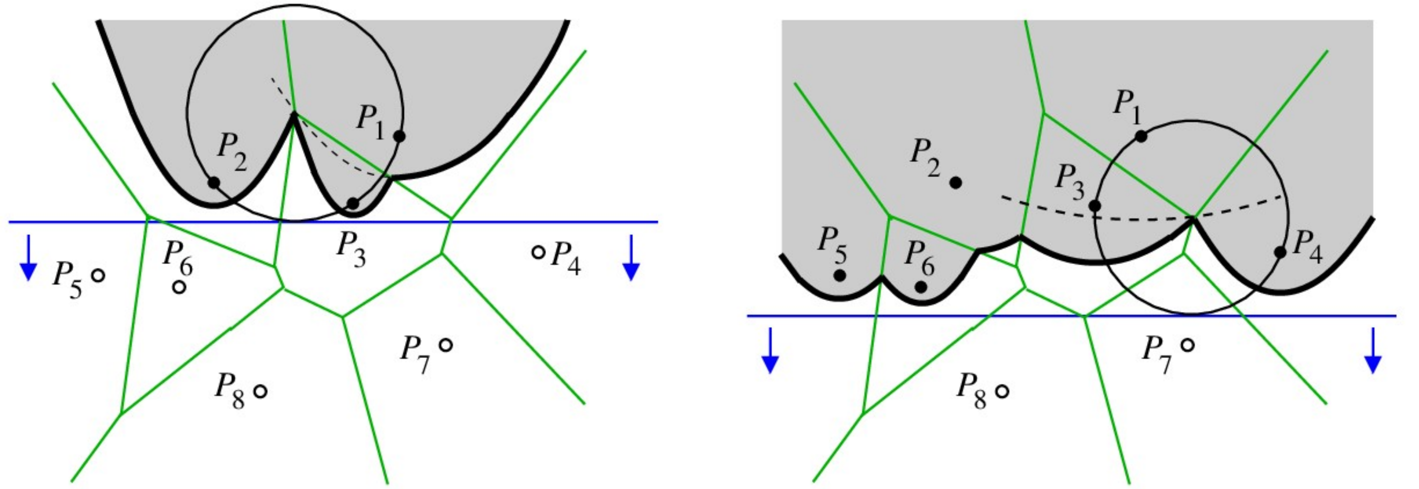
(a) A site event occurs (b) After the site event

P_2, P_1

P_2, P_1, P_3, P_1

P_i, P_j, P_k

vanishes



(a) A fork–out circle event

(b) A fork–in circle event

Position of the sweep line

↳ triggers event

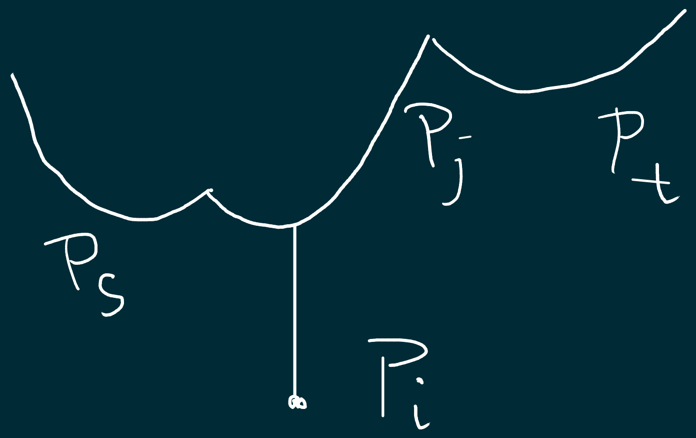
Beach line = A sequence of sites contributing the parabolic arcs.

$$B : (P_3, P_2, P_1, P_2, P_4, P_2, P_5)$$

Event queue Q

- future sites
- circle events (bottom) corresponding to three consecutive arcs on the beach

Handle a site event



Before:

$p_s \; p_j \; p_t$

Q contained the circle event for this triple.

Remove from Q

After

$p_s \; p_j \; p_i \; p_j \; p_t$

insert in Q

Handle a circle event

Before: $P_s \; P_i \; P_j \; P_k \; P_t$

Remove from Q

vanishes

automatically removed by the event handler.

After: $P_s \; P_i \; P_k \; P_t$

Add to Q the circle events for these triples.

Implement both B and Q
as height-balanced BSTs

$O(\log \text{size})$ time $\quad O(\log n)$

First event: sweep line encounters
the top most site P_1

One parabolic arc

Each site event increases the no. of arcs by 2

n site events. The max size of B is $2n-1$

A circle events removes one arc from B.

Max size of $Q = n + 2n - 3 = 3n - 3$

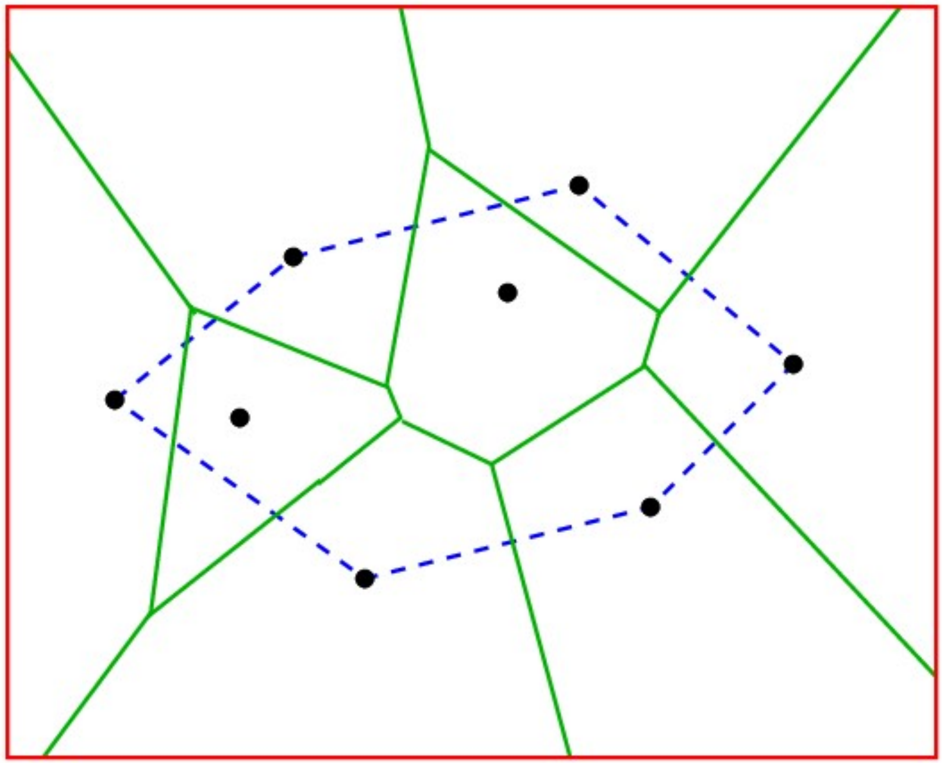
The size of Vor(S) is linear in n.

$O(n)$ circle events.

n site events.

$O(n \log n)$ time

Doubly connected edge list (DCEL)



Handling the semi–infinite lines

Take a bounding box large enough to contain all the sites and all the vertices of Vor(S) discovered.