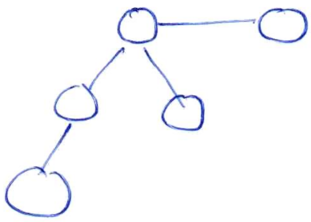
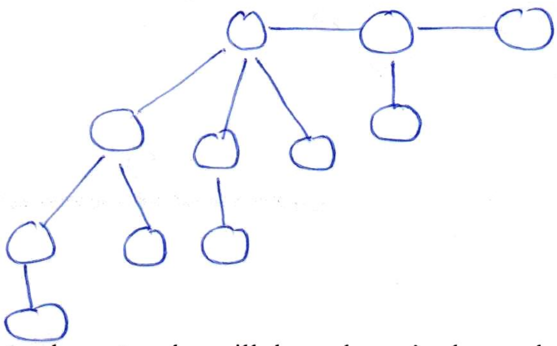
CS31005: Algorithms-II
Class Test 1
Solution Sketch

1. After 1st ExtractMin (5 marks)



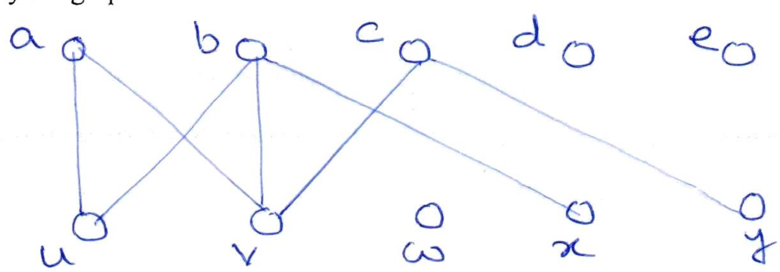
After 2nd ExtractMin (5 marks)



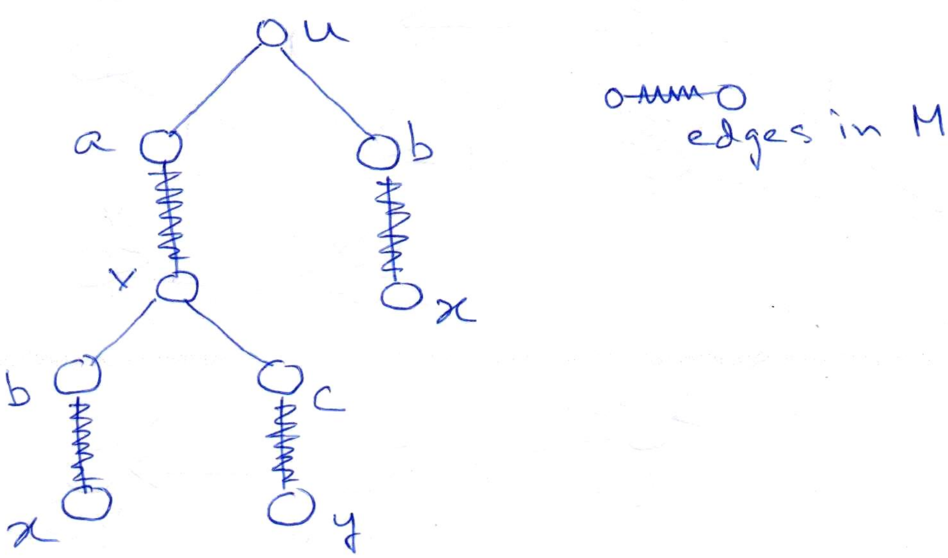
Exact values at nodes will depend on x's chosen, but the structures will look like this (ignoring left/right)

2. Will vary depending on capacities and augmenting paths chosen. Straightforward to do.
Marks distribution: 5 marks for each step (1 for augmenting path, 2 for flow graph after augmentation, 2 for residual graph)

3. Equality subgraph (2 marks)



S = {u, v, x, y}, T = {a, b, c}. The tree is (3 marks)



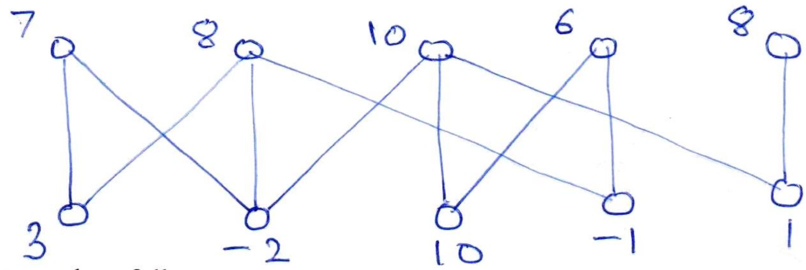
(The repeated (b, x) part can be removed by proper data structure to keep track of paths already seen, shown here for completeness anyway. Given marks for both cases)

To compute α_ℓ, look at edges between S and non-T, including 0-weight edges.

$$\alpha_\ell = \min(8+6-0, 3+6-0, 8+8-0, 3+8-0, 6+4-4, 4+8-0, 5+4-0, 5+8-8) = 5$$

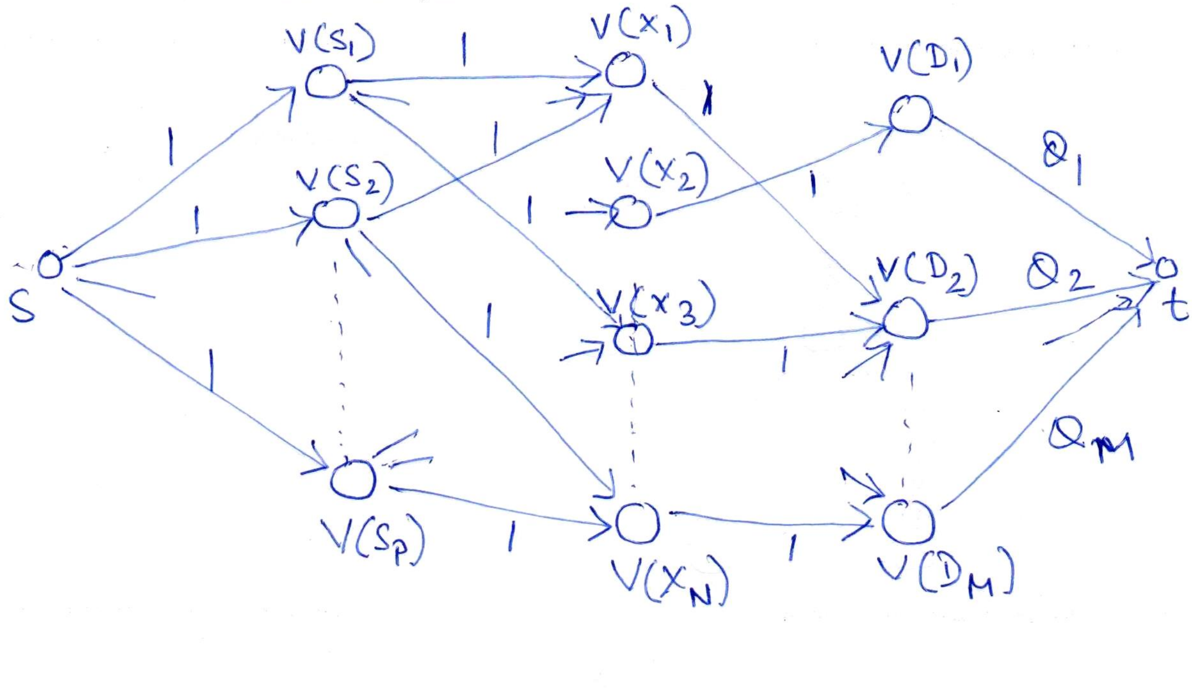
(3 marks)

Labels after relabelling (2 marks)



4. Create a graph as follows:
   - For each society S_i, add a node $V(S_i)$
   - For each student X_j, add a node $V(X_j)$
   - For each department D_k, add a node $V(D_k)$
   - Add a source node s and a sink node t
   - Add an edge from s to each node $V(S_i)$ with capacity 1
   - Add an edge from a node $V(S_i)$ to a node $V(X_j)$ with capacity 1 if the student X_j is a member of society S_i
   - Add an edge from a node $V(X_j)$ to a node $V(D_k)$ with capacity 1 if the student X_j belongs to department D_k
   - Add an edge from each node $V(D_k)$ to t with capacity Q_k

Compute the maximum flow in this graph. If the maximum flow value is = P, the number of societies, then the answer is yes.



Many other variations possible. Note that since we just wanted a yes/no answer and not the actual constitution of the committees, variations are possible with only two layer of nodes also if properly done. All of them have been given credit if correct.

Marks distribution for this is not fixed and varied based on how close to the correct solution your answer is, clarity etc. In general, graded a bit on the tough side.

5. Proof by contradiction. Assume that there exists a path from v to t in the residual graph. Let the path be $v = x_1, x_2, x_3, \ldots., x_p = t$, of length p - 1.
We know that $h(x_1) > k$ and $h(x_p) = 0$.
Consider $j = \text{maximum } \{i \mid h(x_i) > k, 1 \leq i < p\}$, i.e., x_j is the closest node to t in the path with h value > k. Consider the edge (x_j, x_{j+1}).

Then, by property of height function,

$$h(x_j) \leq h(x_{j+1}) + 1$$
$$h(x_{j+1}) \geq h(x_j) - 1$$

This implies that the only possible value of $h(x_{j+1})$ is k, as it cannot be greater than k (as then x_j is not the <u>closest</u> node to t with h value > k), and it cannot be less than k (as then $h(x_j)$ is not greater than k).

This is a contradiction as there does not exist any node u with h(u) = k. Hence the assumption of the existence of the path must be wrong. Hence no path can exist from v to t.

(Marks distribution for this is not fixed and varied based on the logic. In general, many of you have brought in applicability of push/relabel etc., which is totally wrong as the question does not say that any node has any excess.)