CS21003 Algorithms–I, Spring 2017–2018

Class Test 2

13–April–2018 | CSE 107/108/119/120, 07:00pm–08:00pm | Maximum marks: 20

Roll no: ____________ **Name:** ______________________________

[*Write your answers in the question paper itself. Be brief and precise. Answer all questions.*
If you use any algorithm/result/formula covered in the class, just mention it, do not elaborate.]

1. You make a DFS/BFS traversal in a connected undirected graph $G = (V, E)$. For $v \in V$, let $level(v)$ denote the level of v in the DFS/BFS tree T corresponding to your traversal (the root is at level 0, its children are at level 1, the grandchildren of the root are at level 2, and so on). Let $(u, v) \in E$ be a non-tree edge (that is, an edge of G, not belonging to the DFS/BFS tree T).

(a) If the traversal was a DFS traversal, prove that $|level(u) - level(v)| > 1$. **(5)**

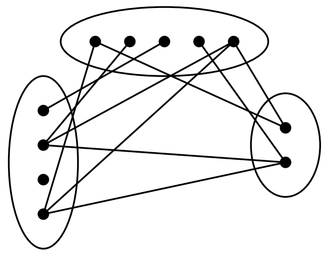
Solution Here, (u, v) is a backward/forward edge. Without loss of generality, assume that $level(u) \leqslant level(v)$, so u is an ancestor of v. If $level(u) = level(v)$, then $u = v$. If $level(u) = level(v) - 1$, then u is the parent of v in T, that is, (u, v) is an edge of T. So we must have $level(u) \leqslant level(v) - 2$.

(b) If the traversal was a BFS traversal, prove that $|level(u) - level(v)| \leqslant 1$. **(5)**

Solution BFS traversal produces shortest distances from the root—call the root r. In particular, for any vertex w, $level(w)$ is the shortest r, w distance (where distances are measured by the numbers of edges on paths). Suppose that $level(v) \geqslant level(u) + 2$. Now, (u, v) is a cross edge. The shortest r, v distance is $level(v)$. However, the r, u-path in T (which is of length $level(u)$) followed by the cross edge (u, v) gives an r, v-path of length $level(u) + 1 \leqslant level(v) - 1 < level(v)$, a contradiction.

2. Recall that an undirected graph $G = (V, E)$ is called bipartite if its vertex set V can be partitioned into two mutually disjoint independent sets V_1 and V_2 (an independent set in a graph is a subset S of vertices such that no two vertices of S share an edge). Likewise, we call G tripartite if its vertex set V can be partitioned into three mutually disjoint independent sets V_1, V_2, V_3. The following figure illustrates a tripartite graph.

(a) Argue that a tripartite graph can have cycles of any length $\geqslant 3$. **(5)**



Solution Let us name the vertices of V_1 as $u_1, u_2, u_3, \ldots$, those of V_2 as $v_1, v_2, v_3, \ldots$, and those of V_3 as $w_1, w_2, w_3, \ldots$. We show that a cycle of any length $l \geqslant 3$ is possible in G.

Solution 1

Case 1: $l = 2k$, $k \geqslant 2$, is even. In this case, we can use only two parts to form a cycle as in bipartite graphs: $(u_1, v_1, u_2, v_2, \ldots, u_k, v_k)$.

Case 2: $l = 2k + 1$, $k \geqslant 1$, is odd. Now, we need to involve the third part. One possibility of a cycle of length $2k + 1$ is $(u_1, v_1, u_2, v_2, \ldots, u_{k-1}, v_{k-1}, u_k, v_k, w_1)$.

Solution 2

Case 1: $l = 3k$, $k \geqslant 1$. $(u_1, v_1, w_1, u_2, v_2, w_2, \ldots, u_k, v_k, w_k)$ can be a cycle in G.

Case 2: $l = 3k + 1$, $k \geqslant 1$. $(u_1, v_1, w_1, u_2, v_2, w_2, \ldots, u_{k-1}, v_{k-1}, w_{k-1}, u_k, v_k, u_{k+1}, v_{k+1})$.

Case 3: $l = 3k + 2$, $k \geqslant 1$. $(u_1, v_1, w_1, u_2, v_2, w_2, \ldots, u_k, v_k, w_k, u_{k+1}, v_{k+1})$.

Solution 3: Induction on *l*

[$l = 3$] Consider a cycle of the form (u_1, v_1, w_1).

[$l > 3$] Let $(x_1, x_2, \ldots, x_{l-1})$ be a cycle of length $l - 1$ in G. Without loss of generality, we can take $x_1 \in V_1$ and $x_{l-1} \in V_2$ (notice that x_{l-1} cannot be in V_1 which is an independent set). We can *extend* the given cycle to the cycle $(x_1, x_2, \ldots, x_{l-1}, x_l)$ of length l by including a *new* vertex x_l from V_3.

(b) Let C_1, C_2, C_3 be three colors. G is called 3-colorable if we can assign these colors to the vertices so that no two adjacent vertices receive the same color. Prove that G is tripartite if and only if G is 3-colorable. **(5)**

Solution [$\Rightarrow$] Let V_1, V_2, V_3 be a tripartition of G. For each $i = 1, 2, 3$, color all the vertices of V_i by C_i. Since each V_i is an independent set, this coloring is proper.

[$\Leftarrow$] Consider any proper 3-coloring of G by the three colors C_1, C_2, C_3. For each $i = 1, 2, 3$, let V_i denote the set of vertices that receive the color C_i. Since each vertex gets a unique color, V_1, V_2, V_3 are mutually disjoint, and $V_1 \cup V_2 \cup V_3 = V$. Moreover, since the 3-coloring was proper, each of V_1, V_2, V_3 must be an independent set.

Remark: Part (b) is a first step to conclude that a polynomial-time algorithm to check whether a graph is tripartite cannot perhaps exist.

FOR LEFTOVER ANSWER AND ROUGH WORK

FOR ROUGH WORK