PDS Lab, Spring 2006

Assignment 7

Let x_0 be an integer and $x_1, x_2, \ldots, x_{k-1}$ positive integers. Consider the expression of the form:

$$x = x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2 + \ddots + \cfrac{1}{x_{k-2} + \cfrac{1}{x_{k-1}}}}}$$

We call the right side of the above equation a *continued fraction expansion* of the (rational number) x. We denote this continued fraction by the short-hand notation

$$x = \langle x_0, x_1, \ldots, x_{k-1} \rangle,$$

and say that the expansion has k terms. It is easy to see that every rational number has such a (finite) continued fraction expansion and that every such (finite) continued fraction expansion stands for a rational number.

An irrational number (like $\sqrt{2}, \pi$) can also be expressed as a continued fraction, but now we have to allow an infinite number of terms. Truncating the expansion after k terms gives a rational approximation of the irrational number. As we consider more and more terms, the approximation becomes better and better.

As an example, take

$$\xi = \pi = 3.14159265358979323846\ldots\ .$$

First set $x_0 = \lfloor \xi \rfloor = 3$, where $\lfloor\ \rfloor$ stands for the integer part. Next replace ξ by

$$\frac{1}{\xi - \lfloor \xi \rfloor} = \frac{1}{0.14159265358979323846\ldots} = 7.06251330593104576992\ldots\ .$$

Then we set $x_1 = \lfloor \xi \rfloor = 7$ and replace ξ by

$$\frac{1}{\xi - \lfloor \xi \rfloor} = \frac{1}{0.06251330593104576992\ldots} = 15.99659440668571985518\ldots\ .$$

Set $x_2 = 15$. This process is repeated as many times as we want, i.e., for $i = 0, 1, 2, \ldots$, we set x_i to be the integer part of the current value of ξ and subsequently replace ξ by $\dfrac{1}{\xi - \lfloor \xi \rfloor}$. The finite continued fraction

$$\xi_k = \langle x_0, x_1, \ldots, x_{k-1} \rangle$$

is called the k-th *convergent* of ξ. It is a rational number and approximates ξ. For the above example, we have

$$\pi = \langle 3, 7, 15, 1, 292, \ldots \rangle,$$

and so the first few convergents are:

$$\begin{array}{lcccccl}
\pi_1 & = & \langle 3 \rangle & = & 3/1 & = & 3.00000000000000000000\ldots\ , \\
\pi_2 & = & \langle 3, 7 \rangle & = & 22/7 & = & 3.14285714285714285714\ldots\ , \\
\pi_3 & = & \langle 3, 7, 15 \rangle & = & 333/106 & = & 3.14150943396226415094\ldots\ , \\
\pi_4 & = & \langle 3, 7, 15, 1 \rangle & = & 355/113 & = & 3.14159292035398230088\ldots\ .
\end{array}$$