CS29003 Algorithms Laboratory

Supplement to Assignment 0

Derivation of $S(i)$

We clearly have $S(0)=1$, $S(1)=3$, and $S(2)=9$. So assume that $i \geqslant 3$. Take $1 \leqslant k \leqslant \lfloor i/3 \rfloor$. Let $\sigma(i,k)$ be the count of all strings of length i containing k occurrences of *abc* and with the remaining $i-3k$ positions filled by arbitrary symbols. Notice that this arbitrary filling may introduce further occurrences of *abc*. For example, *cabcababcabc* is counted thrice in $\sigma(12,1)$, thrice in $\sigma(12,2)$, and once in $\sigma(12,3)$. The principle of inclusion and exclusion implies that

$$S(i) = 3^i - \sigma(i,1) + \sigma(i,2) - \sigma(i,3) + - \ldots + (-1)^j \sigma(i,j),$$

where $j = \lfloor i/3 \rfloor$. For computing $\sigma(i,k)$, think of a string of length i containing k specified occurrences of *abc*. Replace each of these k occurrences of *abc* by a special symbol δ. This gives a string of length $i-2k$ with exactly k occurrences of δ and with $i-3k$ occurrences of a,b,c. For example, consider *cabcababcabc* while calculating $\sigma(12,2)$. If the first and the last occurrences of *abc* are to be taken into account, the string reduces to $c\delta ababc\delta$. The k positions of δ can be specified in $\binom{i-2k}{k}$ ways, and the positions for a,b,c can be filled in 3^{i-3k} ways. Consequently, we have

$$\sigma(i,k) = \binom{i-2k}{k} 3^{i-3k}.$$

We thus get the final formula

$$S(i) = 3^i + \sum_{k=1}^{\lfloor i/3 \rfloor} (-1)^k \binom{i-2k}{k} 3^{i-3k}.$$

The required powers of three can be precomputed and stored in an array in a total of $O(n)$ time. Each binomial coefficient can be computed in $O(n)$ time, so each $S(i)$ can be computed in $O(n^2)$ time using the precomputed powers of three. All the desired $S(i)$ values are therefore computable in a total of $O(n^3)$ time. Eqn (1) subsequently gives $T(n)$ in $O(n)$ time. The overall running time is therefore $O(n^3)$.

Exercise: Reduce the running time of this method to $O(n^2)$.

Hint: We need $\binom{r}{k}$ for $r = 3k, 3k+1, \ldots, n-3$. If some $\binom{r}{k}$ is computed, we have

$$\binom{r+1}{k} = \frac{r+1}{r-k+1} \binom{r}{k}.$$

How does this observation help?