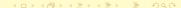
Public-key Cryptography Theory and Practice

Abhijit Das

Department of Computer Science and Engineering Indian Institute of Technology Kharagpur

Chapter 8: Quantum Computation and Cryptography



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- It is not known how to build a quantum computer.
- Some partial implementations are known.



A Disclaimer

There was a time when the newspapers said that only twelve men understood the theory of relativity. I do not believe there ever was such a time . . . On the other hand, I think I can safely say that nobody understands quantum mechanics.

— Richard Feynman (The Character of Physical Law, BBC, 1965)

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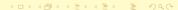
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$$|\psi\rangle=\sum_{i=0}^{2^n-1}a_i|i\rangle$$
 with $a_i\in\mathbb{C}$ and $\sum_{i=0}^{2^n-1}|a_i|^2=1$.

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- Entanglement with surroundings poses the biggest challenge for realizing quantum computers.

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- In particular, all operations on a quantum-mechanical system are invertible.
- **No-cloning theorem:** It is impossible to copy the contents of a quantum register to another. (The transformation $|\psi\rangle|\varphi\rangle\mapsto|\psi\rangle|\psi\rangle$ is not invertible.)



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Hadamard	$H 0\rangle = \frac{1}{\sqrt{2}}(0\rangle + 1\rangle)$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$
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$$H|\psi\rangle = H(a|0\rangle + b|1\rangle)$$

$$= aH|0\rangle + bH|1\rangle$$

$$= a\left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right] + b\left[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right]$$

$$= \left(\frac{a+b}{\sqrt{2}}\right)|0\rangle + \left(\frac{a-b}{\sqrt{2}}\right)|1\rangle$$

$$= (a b)\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}|0\rangle\\|1\rangle\end{pmatrix}$$

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- Measurement is often used to initialize a system.
- So sad! You cannot see Schrödinger's cat in the state $\frac{1}{\sqrt{2}}$ ($|Alive\rangle + |Dead\rangle$).

The Generalized Born Rule

• Let R be an (m+n)-bit quantum register in the state

$$|\psi\rangle_{m+n} = \sum_{i,j} a_{i,j} |i,j\rangle_{m+n}$$
 with $\sum_{i,j} |a_{i,j}|^2 = 1$.

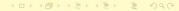
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- We measure the left m bits of R.
- The outcome is an integer $i \in \{0, 1, 2, \dots, 2^m 1\}$ with

probability
$$p_i = \sum_{j=0}^{2^n-1} |a_{i,j}|^2$$
.

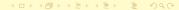
- Let R be an (m+n)-bit quantum register in the state $|\psi\rangle_{m+n}=\sum_{i,j}a_{i,j}|i,j\rangle_{m+n}$ with $\sum_{i,j}\mid a_{i,j}\mid^2=1$.
- We measure the left m bits of R.
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- Probability of measuring $|i\rangle_m|j\rangle_n$ is $p_i \mid a_{i,j} \mid {}^2/p_i = \mid a_{i,j} \mid {}^2.$

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- U_f is a unitary transformation.
- $U_f^{-1} = U_f$.



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- Use a 2-bit register R (m = n = 1).
- Use the unitary transform $D_f|x\rangle|y\rangle=|x\rangle|f(x)\oplus y\rangle$.
- Initialize R to the state $\left(\frac{1}{\sqrt{2}}|0\rangle \frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle \frac{1}{\sqrt{2}}|1\rangle\right)$ = $\frac{1}{2}\left(|0\rangle|0\rangle - |0\rangle|1\rangle - |1\rangle|0\rangle + |1\rangle|1\rangle\right)$.



Applying D_f on R changes its state to

$$\begin{cases} \frac{1}{2} \left(|0\rangle - |1\rangle \right) \left(|f(0)\rangle - |\overline{f}(0)\rangle \right) & \text{if } f(0) = f(1), \\ \frac{1}{2} \left(|0\rangle + |1\rangle \right) \left(|f(0)\rangle - |\overline{f}(0)\rangle \right) & \text{if } f(0) \neq f(1). \end{cases}$$

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- Measure the left bit.
- The outcome is 1 or 0 according as whether f is constant or not.

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Quantum Key Exchange

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- Alice generates a random classical bit i.
- Alice makes a random decision x.
- If x = 0, Alice sends the qubit $|i\rangle$ itself to Bob.
- If x=1, Alice uses the Hadamard transform and sends $H|i\rangle$ $(H|0\rangle=\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$ or $H|1\rangle=\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle))$ to Bob.



Bob processes Alice's qubit

Let A be the qubit received by Bob from Alice.



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Alice and Bob exchange their guesses

- Bob sends y to Alice.
- Alice sends x to Bob.
- If x = y, Alice and Bob store the common bit i = j.



• If x = y = 0, then Alice sends $A = |i\rangle$ to Bob, and Bob measures $B = A = |i\rangle$ to obtain j = i.

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- Now, j gives no clue about i.
- Alice and Bob discard i and j.
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- In about 2*n* iterations, a common *n*-bit key can be established.



Example
Eavesdropping
Practical Implementation

The BB84 Algorithm: Example

Iteration i x A y B j Common bit

Iteration			Α	У	В		Common bit
1	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	
2	0	0	0>	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	

Iteration	i	X	Α	У	В	j	Common bit
1	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	
2	0	0	0>	1	$\frac{1}{\sqrt{2}}(0\rangle + 1\rangle)$	1	
3	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	0>	0	0

Iteration	i	X	Α	У	В	j	Common bit
1	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	
2	0	0	0>	1	$\frac{\sqrt{1}}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	
3	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	0>	0	0
4	1	1	$\frac{1}{\sqrt{2}}(0\rangle + 1\rangle)$ $\frac{1}{\sqrt{2}}(0\rangle - 1\rangle)$	0	$\frac{1}{\sqrt{2}}(\ket{0}-\ket{1})$	1	

Iteration	i	X	Α	У	В	j	Common bit
1	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	
2	0	0	0>	1	$\frac{\sqrt{12}}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	
3	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	0>	0	0
4	1	1	$\frac{1}{\sqrt{2}}(0\rangle - 1\rangle)$	0	$rac{1}{\sqrt{2}}(\ket{0}-\ket{1})$	1	
5	0	0	0>	0	0>	0	0

Iteration	i	X	Α	У	В	j	Common bit
1	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	1/2 (1-/ ' 1-//	1	
2	0	0	0>	1	$\frac{\sqrt{12}}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	
3	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	0>	0	0
4	1	1	$\frac{\sqrt{12}}{\sqrt{2}}(0\rangle- 1\rangle)$	0	$\frac{1}{\sqrt{2}}(0\rangle- 1\rangle)$	1	
5	0	0	0>	0	0>	0	0
6	1	1	$\frac{1}{\sqrt{2}}(0\rangle- 1\rangle)$	1	1 angle	1	1

Iteration	i	X	Α	У	В	j	Common bit
1	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	
2	0	0	0>	1	$rac{1}{\sqrt{2}}(\ket{0}+\ket{1}) \ rac{1}{\sqrt{2}}(\ket{0}+\ket{1})$	1	
3	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	0>	0	0
4	1	1	$\frac{\sqrt{12}}{\sqrt{2}}(0\rangle- 1\rangle)$	0	$\frac{1}{\sqrt{2}}(0\rangle- 1\rangle)$	1	
5	0	0	0>	0	0>	0	0
6	1	1	$\frac{1}{\sqrt{2}}(0\rangle- 1\rangle)$	1	1 angle	1	1
7	1	1	$\frac{\sqrt{1^2}}{\sqrt{2}}(0\rangle- 1\rangle)$	0	$rac{1}{\sqrt{2}}(\ket{0}-\ket{1})$	0	

Iteration	i	X	Α	У	В	j	Common bit
1	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	
2	0	0	0>	1	$\frac{\sqrt{12}}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	
3	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	0>	0	0
4	1	1	$\frac{\sqrt{2}}{\sqrt{2}}(0\rangle- 1\rangle)$	0	$\frac{1}{\sqrt{2}}(0\rangle- 1\rangle)$	1	
5	0	0	0>	0	0>	0	0
6	1	1	$\frac{1}{\sqrt{2}}(0\rangle - 1\rangle)$	1	1 angle	1	1
7	1	1	$\frac{\sqrt{1}}{\sqrt{2}}(0\rangle - 1\rangle)$	0	$\frac{1}{\sqrt{2}}(\ket{0}-\ket{1})$	0	
8	0	0	0>	0	0>	0	0

Iteration	i	Χ	Α	У	В	j	Common bit
1	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	
2	0	0	0>	1	$\frac{\sqrt{12}}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	
3	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	$ 0\rangle$	0	0
4	1	1	$\frac{\sqrt{2}}{\sqrt{2}}(0\rangle - 1\rangle)$	0	$\frac{1}{\sqrt{2}}(0\rangle - 1\rangle)$	1	
5	0	0	0>	0	0>	0	0
6	1	1	$\frac{1}{\sqrt{2}}(0\rangle - 1\rangle)$	1	1 angle	1	1
7	1	1	$\frac{\sqrt{1-}}{\sqrt{2}}(0\rangle- 1\rangle)$	0	$\frac{1}{\sqrt{2}}(0\rangle- 1\rangle)$	0	
8	0	0	0>	0	0>	0	0
9	1	0	$ 1\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle- 1\rangle)$	1	

Iteration	i	Х	Α	У	В	j	Common bit
1	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	
2	0	0	0>	1	$\frac{\sqrt{2}}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	
3	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	0>	0	0
4	1	1	$\frac{\frac{v_1^2}{\sqrt{2}}}{(0\rangle - 1\rangle)}$	0	$\frac{1}{\sqrt{2}}(0\rangle- 1\rangle)$	1	
5	0	0	0>	0	0>	0	0
6	1	1	$\frac{1}{\sqrt{2}}(0\rangle - 1\rangle)$	1	$ 1\rangle$	1	1
7	1	1	$\frac{\sqrt{1}}{\sqrt{2}}(0\rangle - 1\rangle)$	0	$\frac{1}{\sqrt{2}}(\ket{0}-\ket{1})$	0	
8	0	0	0>	0	0>	0	0
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3	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	0>	0	0
4	1	1	$\frac{\sqrt{12}}{\sqrt{2}}(0\rangle- 1\rangle)$	0	$\frac{1}{\sqrt{2}}(0\rangle- 1\rangle)$	1	
5	0	0	0>	0	0>	0	0
6	1	1	$\frac{1}{\sqrt{2}}(0\rangle - 1\rangle)$	1	$ 1\rangle$	1	1
7	1	1	$\frac{\sqrt{1^2}}{\sqrt{2}}(0\rangle- 1\rangle)$	0	$rac{1}{\sqrt{2}}(\ket{0}-\ket{1})$	0	
8	0	0	0>	0	0>	0	0
9	1	0	1 angle	1	$\frac{1}{\sqrt{2}}(0\rangle- 1\rangle)$	1	
10	1	0	$ 1\rangle$	0	1>	1	1
11	0	0	$ 0\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	



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1	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	
2	0	0	0>	1	$rac{1}{\sqrt{2}}(\ket{0}+\ket{1}) \ rac{1}{\sqrt{2}}(\ket{0}+\ket{1})$	1	
3	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	0>	0	0
4	1	1	$\frac{\sqrt{12}}{\sqrt{2}}(0\rangle - 1\rangle)$	0	$\frac{1}{\sqrt{2}}(\ket{0}-\ket{1})$	1	
5	0	0	0>	0	0>	0	0
6	1	1	$\frac{1}{\sqrt{2}}(0\rangle- 1\rangle)$	1	$ 1\rangle$	1	1
7	1	1	$\frac{\sqrt{1}^2}{\sqrt{2}}(0\rangle - 1\rangle)$	0	$rac{1}{\sqrt{2}}(\ket{0}-\ket{1})$	0	
8	0	0	0>	0	0>	0	0
9	1	0	1 angle	1	$\frac{1}{\sqrt{2}}(0\rangle- 1\rangle)$	1	
10	1	0	1 angle	0	1>	1	1
11	0	0	$ 0\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	
12	0	0	$ 0\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	

Iteration	i	X	Α	У	В	j	Common bit
1	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	
2	0	0	0>	1	$rac{1}{\sqrt{2}}(\ket{0}+\ket{1})$ $rac{1}{\sqrt{2}}(\ket{0}+\ket{1})$	1	
3	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	0>	0	0
4	1	1	$\frac{\frac{1}{\sqrt{2}}(0\rangle- 1\rangle)}{\frac{1}{\sqrt{2}}}$	0	$\frac{1}{\sqrt{2}}(0\rangle- 1\rangle)$	1	
5	0	0	0>	0	0>	0	0
6	1	1	$\frac{1}{\sqrt{2}}(0\rangle - 1\rangle)$	1	1 angle	1	1
7	1	1	$\frac{\frac{1}{\sqrt{2}}(0\rangle - 1\rangle)}{\frac{1}{\sqrt{2}}(0\rangle - 1\rangle)}$	0	$\frac{1}{\sqrt{2}}(\ket{0}-\ket{1})$	0	
8	0	0	0>	0	0>	0	0
9	1	0	1 angle	1	$\frac{1}{\sqrt{2}}(0\rangle - 1\rangle)$	1	
10	1	0	1 angle	0	1>	1	1
11	0	0	$ 0\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	
12	0	0	$ 0\rangle$	1	$\frac{\sqrt{12}}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	
13	1	1	$rac{1}{\sqrt{2}}(\ket{0}-\ket{1})$	1	1>	1	1

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- If x = y, Alice stores i, and Bob stores j.
- Carol may have caused $i \neq j$ even when x = y.



Iter i x A z $C = H^z A$ k D y $B = H^y D$ j

Iter	i	X	Α	Z	$C = H^z A$	k	D	У	$B = H^y D$	j
1	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	0⟩	0	0>	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0
2	1	0	1>	0	1⟩	1	$ 1\rangle$	0	1>	1

					$C = H^z A$					
1	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	0⟩	0	0>	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0
2	1	0			$ 1\rangle$					
3	1	0	1 angle	1	$\frac{1}{\sqrt{2}}(0\rangle- 1\rangle)$	0	$ 0\rangle$	0	$ 0\rangle$	0

Iter	i	X	Α	Z	$C = H^z A$	k	D	У	$B = H^y D$	j
1	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	0⟩	0	0>	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0
2	1	0	1>	0	$ 1\rangle$	1	$ 1\rangle$	0	1>	1
					$\frac{1}{\sqrt{2}}(0\rangle- 1\rangle)$					0
4	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	$ 0\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1

Iter	i	X	Α	Z	$C = H^z A$	k	D	У	$B = H^y D$	j
1	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	0⟩	0	0>	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0
					$ 1\rangle$				1>	1
3	1	0	1 angle	1	$\frac{1}{\sqrt{2}}(0\rangle- 1\rangle)$	0	$ 0\rangle$	0	$ 0\rangle$	0
4	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	$ 0\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1
									$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	

	i								$B = H^y D$	j
1	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	0⟩	0	0>	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0
	1		1>						1>	1
3	1	0	$ 1\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle- 1\rangle)$	0	$ 0\rangle$	0	$ 0\rangle$	0
4	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	$ 0\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1
5					0⟩			1	4	1
6	1	1	$\frac{1}{\sqrt{2}}(0\rangle - 1\rangle)$	1	$ 1\rangle$	1	$ 1\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle - 1\rangle)$	1

Iter	i	X	Α				D	У	$B = H^y D$	j
1	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	0⟩	0	0>	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0
2	1	0		0	$ 1\rangle$		$ 1\rangle$	0	1>	1
3	1	0	1 angle	1	$\frac{1}{\sqrt{2}}(0\rangle- 1\rangle)$	0	$ 0\rangle$	0	$ 0\rangle$	0
4	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	$ 0\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1
5	0	1			0>		$ 0\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1
6	1	1	$\frac{1}{\sqrt{2}}(0\rangle - 1\rangle)$	1	$ 1\rangle$	1	$ 1\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle - 1\rangle)$	1
7	1	1				0		1	• =	0

Iter	i	X	Α	Z	$C = H^z A$	k	D	У	$B = H^y D$	j
1	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	0⟩	0	0>	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0
2	1	0	1>	0	$ 1\rangle$	1	$ 1\rangle$	0	1>	1
3	1	0	1 angle	1	$\frac{1}{\sqrt{2}}(0\rangle- 1\rangle)$	0	$ 0\rangle$	0	$ 0\rangle$	0
4	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	$ 0\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1
5	0	1	1/2 (1 / 1 / /	1	$ 0\rangle$	0	$ 0\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1
6	1	1		1	1⟩	1	$ 1\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle - 1\rangle)$	1
7	1	1	$\frac{1}{\sqrt{2}}(0\rangle - 1\rangle)$	0	$\frac{1}{\sqrt{2}}(0\rangle- 1\rangle)$	0	$ 0\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0
8	1	0	1>	0	1>	1	$ 1\rangle$	0	1>	1

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1	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	0⟩	0	0>	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0
2	1	0	1>	0	$ 1\rangle$	1	$ 1\rangle$	0	1>	1
3	1	0	$ 1\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle- 1\rangle)$	0	$ 0\rangle$	0	$ 0\rangle$	0
4	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	$ 0\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1
5	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	0>	0	$ 0\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1
6	1	1		1	$ 1\rangle$	1	$ 1\rangle$	1	$\frac{\sqrt{1}}{\sqrt{2}}(0\rangle - 1\rangle)$	1
7	1	1	$\frac{1}{\sqrt{2}}(0\rangle- 1\rangle)$	0	$\frac{1}{\sqrt{2}}(0\rangle- 1\rangle)$	0	$ 0\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0
8	1	0	1>	0	1>	1	$ 1\rangle$	0	1>	1
9	1	1	$\frac{1}{\sqrt{2}}(\ket{0}-\ket{1})$	0	$\frac{1}{\sqrt{2}}(0\rangle- 1\rangle)$	0	$ 0\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1

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2	1	0	1>	0	$ 1\rangle$	1	$ 1\rangle$	0	1>	1
3	1	0	$ 1\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle- 1\rangle)$	0	$ 0\rangle$	0	$ 0\rangle$	0
4	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	$ 0\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1
5	0	1	$\sqrt{2}$ (1 / · 1 / /	1	$ 0\rangle$	0	$ 0\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1
6	1	1	$\frac{1}{\sqrt{2}}(0\rangle - 1\rangle)$	1	1>	1	$ 1\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle - 1\rangle)$	1
7	1	1	$\frac{1}{\sqrt{2}}(0\rangle - 1\rangle)$	0	$\frac{1}{\sqrt{2}}(0\rangle- 1\rangle)$	0	$ 0\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0
8	1	0	1>	0	1>	1	$ 1\rangle$	0	1>	1
9	1	1	$\frac{1}{\sqrt{2}}(0\rangle - 1\rangle)$	0	$\frac{1}{\sqrt{2}}(0\rangle - 1\rangle)$	0	$ 0\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1
10	0	1	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	0	$\frac{1}{\sqrt{2}}(0\rangle+ 1\rangle)$	1	$ 1\rangle$	1	$\frac{1}{\sqrt{2}}(0\rangle- 1\rangle)$	1

i and j differ in five positions.



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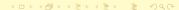
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- If eavesdropping is detected, the key exchange session is discarded.



Polarization angle	Qubit value
00	0>
45 ⁰	$\frac{1}{\sqrt{2}}(0\rangle + 1\rangle)$
90 ⁰	1>
135 ⁰	$\frac{1}{\sqrt{2}}(0\rangle - 1\rangle)$
	V Z

Polarization of photons can be used.

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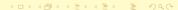
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- If not (or if r is odd), repeat with another a.





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- The classical algorithm may take exponential time (in log *m*).
- Shor's algorithm computes r with high probability by making only a single evaluation of f.



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- Having a collision $x_u = x_v$ is governed by the birthday paradox, and the algorithm becomes exponential again.

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- The probability that we measure a useful y is at least $\frac{4}{x^2} = 0.40528...$
- If the measured y is useful, we run a classical algorithm (based upon continued fractions) to obtain a factor of r.

