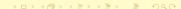
Public-key Cryptography Theory and Practice

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Chapter 4: The Intractable Mathematical Problems



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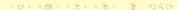
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- Certain special cases have been discovered to be cryptographically weak. For practical designs, it is essential to avoid these special cases.
- Polynomial-time quantum algorithms are known for factoring integers and computing discrete logarithms in finite fields.

• Let \mathbb{F}_q be a finite field, g a generator of \mathbb{F}_q^* , and $a \in \mathbb{F}_q^*$. There exists a unique integer $x \in \{0, 1, 2, \dots, q-1\}$ such that $a = g^x$. We call x the *index* or *discrete logarithm* of a to the base g. We denote this by $x = \operatorname{ind}_g a$.

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 The concept of discrete logarithms can be extended to other finite groups (including the elliptic curve group).

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• $ind_3 6 = 15$ and $ind_3 11 = 7$. Since $6 \times 11 = 15 \pmod{17}$, we have $ind_3 15 \equiv ind_3 6 + ind_3 11 \equiv 15 + 7 \equiv 6 \pmod{16}$.

Integer factorization problem (IFP): Given $n \in \mathbb{N}$, compute the complete prime factorization of n. Suppose there is an algorithm A that computes a non-trivial factor of n. We can use A repeatedly in order to compute the complete factorization of n. If n = pq (with $p, q \in \mathbb{P}$), then computing p or q suffices.

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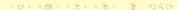
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- The converse is only believed to be true.



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- Let $Q_1 = 3P = (5,0)$ and $Q_2 = 4P = (6,1)$. Then, $(3 \times 4)P = 12P = 0P = \mathcal{O}$.



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- Computing a generator of \mathbb{F}_q^* requires the complete factorization of q-1.
- Generalized Discrete Logarithm Problem (GDLP) Let G be a (multiplicative) Abelian group of size n and let g be an element of G of order m (we have $m \mid n$). Let H be the subgroup of G generated by g. Given $a \in G$, determine whether $a \in H$, and if so, determine the unique integer $x \in \{0, 1, 2, \ldots, m-1\}$ such that $a = g^x$.

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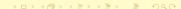
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In the worst case, these algorithms take exponential (in log *n*) running time.



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Algorithm	Inventor(s)	Running time
Continued fraction method (CFRAC)	Morrison & Brillhart (1975)	L(n,1/2,c)
Quadratic sieve method (QSM)	Pomerance (1984)	L(n,1/2,1)
Cubic sieve method (CSM)	Reyneri	L(n, 1/2, 0.816)
Elliptic curve method (ECM)	H. W. Lenstra (1987)	L(n, 1/2, c)
Number field sieve method (NFSM)	A. K. Lenstra, H. W. Lenstra, Manasse & Pollard (1990)	<i>L</i> (<i>n</i> , 1/3, 1.923)

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 - Thus, we have $n = 7^2 \times 53 \times 661$.
- Trial division is efficient if n has only small prime factors (except possibly one).

Let $p \leqslant \sqrt{n}$ be an unknown prime divisor of n. Let $f : \mathbb{Z}_n \to \mathbb{Z}_n$ be a pseudorandom function.

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- By the **Birthday Paradox**, the expected running time of Pollard's rho method is $O^{\sim}(\sqrt[4]{n})$.



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- The x sequence does not show the same periodicity.
- $\gcd(61986 48778, n) = 127.$

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To find integers $x, y \in \mathbb{Z}_n$ such that $x^2 \equiv y^2 \pmod{n}$. Unless $x \equiv \pm y \pmod{n}$, $\gcd(x - y, n)$ is a non-trivial divisor of n.

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If n is composite (but not a prime power), then for a randomly chosen pair (x, y) with $x^2 \equiv y^2 \pmod{n}$, the probability that $x \not\equiv \pm y \pmod{n}$ is at least 1/2.

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Suppose T(c) factors over small primes p_1, p_2, \dots, p_t :

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The left side is already a square.

The right side is also a square if each α_i is even.

But this is very rare.



Collect many relations:

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Let
$$\beta_1, \beta_2, \dots, \beta_r \in \{0, 1\}.$$

$$\left[(H+c_1)^{\beta_1}(H+c_2)^{\beta_2}\cdots(H+c_r)^{\beta_r}\right]^2\equiv p_1^{\gamma_1}p_2^{\gamma_2}\cdots p_t^{\gamma_t}\ (\mathrm{mod}\ n).$$

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Tune $\beta_1, \beta_2, \dots, \beta_r$ to make each γ_i even.



$$\alpha_{11}\beta_{1} + \alpha_{21}\beta_{2} + \dots + \alpha_{r1}\beta_{r} = \gamma_{1},$$

$$\alpha_{12}\beta_{1} + \alpha_{22}\beta_{2} + \dots + \alpha_{r2}\beta_{r} = \gamma_{2},$$

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Linear system with t equations and r variables $\beta_1, \beta_2, \ldots, \beta_r$:

$$\begin{pmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{r1} \\ \alpha_{12} & \alpha_{22} & \cdots & \alpha_{r2} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha_{1t} & \alpha_{2t} & \cdots & \alpha_{rt} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_t \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \pmod{2}.$$

For $r \ge t$, there are non-zero solutions for $\beta_1, \beta_2, \dots, \beta_r$. Take

$$x \equiv (H + c_1)^{\beta_1} (H + c_2)^{\beta_2} \cdots (H + c_r)^{\beta_r} \pmod{n},$$

$$y \equiv p_1^{\gamma_1/2} p_2^{\gamma_2/2} \cdots p_t^{\gamma_t/2} \pmod{n}.$$

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$$y \equiv \rho_1^{\gamma_1/2} \rho_2^{\gamma_2/2} \cdots \rho_t^{\gamma_t/2} \pmod{n}.$$

If $x \not\equiv \pm y \pmod{n}$, then $\gcd(x - y, n)$ is a non-trivial factor of n.

Let $p = p_i$ be a small prime.

$$p \mid T(c)$$
 implies $(H + c)^2 \equiv n \pmod{p}$.

If *n* is not a quadratic residue modulo p, then $p \nmid T(c)$ for any c.

Consider only the small primes p modulo which n is a quadratic residue.



Example of QSM: Parameters

$$n = 7116491$$
.

$$H = \lceil \sqrt{n} \rceil = 2668.$$

Take all primes < 100 modulo which n is a square:

$$B = \{2, 5, 7, 17, 29, 31, 41, 59, 61, 67, 71, 79, 97\}.$$

$$t = 13$$
.

Take r = 13. (In practice, one takes $r \approx 2t$.)

Example of QSM: Relations

```
(H+3)^2 \equiv 2 \times 5^3 \times 71
 Relation 1:
                    (H+8)^2 \equiv 5 \times 7 \times 31 \times 41
 Relation 2:
 Relation 3: (H + 49)^2 \equiv 2 \times 41^2 \times 79
 Relation 4: (H + 64)^2 \equiv 7 \times 29^2 \times 59
                 (H+81)^2 \equiv 2\times5\times7^2\times29\times31
 Relation 5:
                  (H+109)^2
                                   \equiv 2 \times 7 \times 17 \times 41 \times 61
 Relation 6:
                                   \equiv 5<sup>3</sup> × 71 × 79
                 (H+128)^2
 Relation 7:
                                                                           \pmod{n}.
                                   \equiv 2 \times 71^2 \times 79
 Relation 8:
                  (H+145)^2
                                   \equiv 17^2 \times 59^2
                  (H+182)^2
 Relation 9:
                                   \equiv 5<sup>2</sup> × 7<sup>2</sup> × 17 × 61
Relation 10: (H + 228)^2
                                   \equiv 2 \times 7^2 \times 17 \times 29 \times 31
Relation 11:
                 (H + 267)^2
Relation 12: (H + 382)^2 \equiv 7 \times 59 \times 67 \times 79
Relation 13: (H + 411)^2
                                   \equiv 2 \times 5^4 \times 31 \times 61
```

Example of QSM: Linear System

(mod 2).

Example of QSM: Solution of Relations

$(\beta_1,\beta_2,\beta_3,\ldots,\beta_{13})$	Χ	У	gcd(x-y,n)
(0,0,0,0,0,0,0,0,0,0,0,0,0,0)	1	1	7116491
(1,0,1,0,0,0,1,0,0,0,0,0,0)	1755331	560322	1847
(0,0,1,0,0,0,0,1,0,0,0,0,0)	526430	459938	1847
(1,0,0,0,0,0,1,1,0,0,0,0,0)	7045367	7045367	7116491
(0,0,0,0,0,0,0,0,1,0,0,0,0)	2850	1003	1847
(1,0,1,0,0,0,1,0,1,0,0,0,0)	6916668	6916668	7116491
(0,0,1,0,0,0,0,1,1,0,0,0,0)	5862390	5862390	7116491
(1,0,0,0,0,0,1,1,1,0,0,0,0)	3674839	6944029	1847
(0,1,0,0,1,1,0,0,0,0,1,0,1)	1079130	3965027	3853
(1,1,1,0,1,1,1,0,0,0,1,0,1)	5466596	1649895	1
(0,1,1,0,1,1,0,1,0,0,1,0,1)	5395334	1721157	1
(1,1,0,0,1,1,1,1,0,0,1,0,1)	6429806	3725000	3853
(0,1,0,0,1,1,0,0,1,0,1,0,1)	1196388	5920103	1
(1,1,1,0,1,1,1,0,1,0,1,0,1)	1799801	3818773	3853
(0,1,1,0,1,1,0,1,1,0,1,0,1)	5081340	4129649	3853
(1,1,0,0,1,1,1,1,1,0,1,0,1)	7099266	17225	1



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- For each solution χ, subtract log q from A_c for all c ≡ χ (mod q^h).

- To identify which values of $T(c) = J + 2Hc + c^2$ are smooth with respect to the factor base B.
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- Each smooth T(c) is factored by trial division.

The Baby-Step-Giant-Step (BSGS) Method ndex Calculus Method for Prime Fields ndex Calculus Method for Fields of Characteristic 2

The Discrete Logarithm Problem

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To compute the discrete logarithm of a in \mathbb{F}_q^* to the primitive base g.

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Older algorithms

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Worst-case complexity: Exponential in log q



The Baby-Step-Giant-Step (BSGS) Method Index Calculus Method for Prime Fields Index Calculus Method for Fields of Characteristic 2

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Modern algorithms

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Algorithm	Inventor(s)	Running time
Basic ICM	Western & Miller (1968)	L(q, 1/2, c)
Linear sieve method (LSM)	Coppersmith, Odlyzko	
Residue list sieve method	& Schroeppel (1986)	L(q, 1/2, 1)
Gaussian integer method		
Cubic sieve method (CSM)	Reyneri	L(q, 1/2, 0.816)
Number field sieve method	Gordon (1993)	L(q, 1/3, 1.923)
(NFSM) [for $\mathbb{F}_{ ho}$ only]		
Coppersmith's method	Coppersmith	L(q, 1/3, 1.526)
[for \mathbb{F}_{2^n} only]		

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We plan to compute $ind_a(g)$ for some $a \in G$.

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- **Giant steps:** For j = 0, 1, 2, ..., m 1, compute ag^{-jm} , and try to locate ag^{-jm} in the table of baby steps.
- If a search is successful, we have $ag^{-jm} = g^i$ for some i, j, that is, $a = g^{jm+i}$, that is, $\operatorname{ind}_g(a) = jm + i$.

Take
$$G = \mathbb{F}_{43}^*$$
 with size $n = 42$, $m = \left\lceil \sqrt{42} \right\rceil = 7$, and $g = 19$.

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Giant Steps: Take a = 3.

• j = 0: $ag^{-0m} \equiv 3 \pmod{43}$ is not in the table.

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Table of Baby Steps

- j = 0: $ag^{-0m} \equiv 3 \pmod{43}$ is not in the table.
- j = 1: $ag^{-m} \equiv 21 \pmod{43}$ is not in the table.

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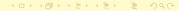
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- j = 4: $ag^{-4m} \equiv 22 \equiv g^3 \pmod{43}$, so $\operatorname{ind}_g(a) = 4 \times 7 + 3 = 31$.



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Linear equation in t variables d_1, d_2, \ldots, d_t :

$$j \equiv \alpha_1 d_1 + \alpha_2 d_2 + \cdots + \alpha_t d_t \pmod{p-1}$$

The Basic ICM: Precomputation (contd)

Generate $r \ge t$ relations for different values of j:

Relation 1:
$$j_1 \equiv \alpha_{11}d_1 + \alpha_{12}d_2 + \cdots + \alpha_{1t}d_t$$

Relation 2: $j_2 \equiv \alpha_{21}d_1 + \alpha_{22}d_2 + \cdots + \alpha_{2t}d_t$
 \cdots
Relation r : $j_r \equiv \alpha_{r1}d_1 + \alpha_{r2}d_2 + \cdots + \alpha_{rt}d_t$ (mod $p-1$).

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Solve the system modulo p-1 to determine d_1, d_2, \ldots, d_t .

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Take discrete log:

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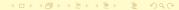
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Take discrete log:

$$\operatorname{ind}_{g} a \equiv -j + \beta_{1} d_{1} + \beta_{2} d_{2} + \cdots + \beta_{t} d_{t} \pmod{p-1}.$$

Substitute the values of d_1, d_2, \dots, d_t to get $ind_g a$.



The Basic ICM: Example (Precomputation)

Parameters: p = 839, g = 31, $B = \{2, 3, 5, 7, 11\}$, t = 5, r = 10.

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Relations

```
g^{118} \equiv 2^3 \times 5^2
Relation 1:
                   q^{574} \equiv 2^7 \times 5
Relation 2:
                   q^{318} \equiv 2^2 \times 3^3
Relation 3:
                   g^{46} \equiv 2^7
Relation 4:
                   q^{786} \equiv 2^2 \times 3^3 \times 7
Relation 5:
                                                        \pmod{p}.
                   q^{323} \equiv 2 \times 3 \times 11
Relation 6:
                   q^{606} \equiv 3^4
Relation 7:
                   q^{252} \equiv 2^3 \times 3^2 \times 7
Relation 8:
                   q^{160} \equiv 3 \times 5^2
Relation 9:
                    q^{600} \equiv 2 \times 3^3 \times 5
Relation 10:
```

The Basic ICM: Example (Precomputation)

$$\begin{pmatrix} 3 & 0 & 2 & 0 & 0 \\ 7 & 0 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 4 & 0 & 0 & 0 \\ 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{pmatrix} \equiv \begin{pmatrix} 118 \\ 574 \\ 318 \\ 46 \\ 786 \\ 323 \\ 606 \\ 252 \\ 160 \\ 600 \end{pmatrix} \pmod{p-1}.$$

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$$\begin{array}{llll} d_1 & \equiv & \operatorname{ind}_{31} \, 2 & = & 246 \\ d_2 & \equiv & \operatorname{ind}_{31} \, 3 & = & 780 \\ d_3 & \equiv & \operatorname{ind}_{31} \, 5 & = & 528 \\ d_4 & \equiv & \operatorname{ind}_{31} \, 7 & = & 468 \\ d_5 & \equiv & \operatorname{ind}_{31} \, 11 & = & 135 \end{array} \right\} \pmod{p-1}.$$

The Baby-Step-Giant-Step (BSGS) Method Index Calculus Method for Prime Fields Index Calculus Method for Fields of Characteristic 2

The Basic ICM: Example (Second Stage)

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Take a = 561.

$$ag^{312} \equiv 600 \equiv 2^3 \times 3 \times 5^2 \pmod{p}$$
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Take a = 625.

$$ag^{806} \equiv 70 \equiv 2 \times 5 \times 7 \pmod{p}$$
, that is, ind₃₁ 625 $\equiv -806 + 246 + 528 + 468 \equiv 436 \pmod{p-1}$.



Represent $\mathbb{F}_{2^n} = \mathbb{F}_2(\alpha)$, where $f(\alpha) = 0$. Let $g(\alpha)$ be a generator of $\mathbb{F}_{2^n}^*$. We plan to compute $\operatorname{ind}_{g(\alpha)} t(\alpha)$.

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- **Linear algebra:** Solve the resulting system of congruences modulo $2^n 1$, and obtain the indices $\operatorname{ind}_{a(\alpha)} u(\alpha)$ for all $u(\alpha) \in B$.
- Second stage: Generate a single relation of the form:

$$t(\alpha)g(\alpha)^{j}=\prod_{u(\alpha)\in B}u(\alpha)^{\delta_{u(\alpha)}}.$$

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- Relations in the first stage

$$g(\alpha)^{7} = \alpha^{6} + \alpha^{2} = \alpha^{2}(\alpha + 1)^{4},$$

$$g(\alpha)^{101} = \alpha^{4} + \alpha^{3} + \alpha + 1 = (\alpha + 1)^{2}(\alpha^{2} + \alpha + 1),$$

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Linear system of congruences

$$\begin{pmatrix} 2 & 4 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} d_{\alpha} \\ d_{\alpha+1} \\ d_{\alpha^2+\alpha+1} \end{pmatrix} \equiv \begin{pmatrix} 7 \\ 101 \\ 121 \end{pmatrix} \pmod{127},$$

where
$$d_{\beta} = \operatorname{ind}_{q(\alpha)}(\beta)$$
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Indices of factor base elements

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$$t(\alpha) = \alpha^3 + 1$$
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• $t(\alpha)g(\alpha)^{57} = \alpha^5 + \alpha^3 = \alpha^3(\alpha + 1)^2$.
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 - $t(\alpha) = \alpha^6 + \alpha^5 + \alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1$. • $t(\alpha)g(\alpha)^{73} = \alpha^5 + \alpha^4\alpha^2 + \alpha = \alpha(\alpha + 1)^2(\alpha^2 + \alpha + 1)$. • $ind_{g(\alpha)}t(\alpha) \equiv -73 + d_{\alpha} + 2d_{\alpha+1} + d_{\alpha^2+\alpha+1} \equiv 24 \pmod{127}$.

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- $t(\alpha) = \alpha^5 + 1$. $t(\alpha)g(\alpha)^{18} = \alpha^5 + \alpha^3 + \alpha = \alpha(\alpha^2 + \alpha + 1)^2$. $ind_{g(\alpha)} t(\alpha) \equiv -18 + d_{\alpha} + 2d_{\alpha^2 + \alpha + 1} \equiv 38 \pmod{127}$.



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Solving Large Sparse Linear Systems

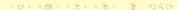
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• **Iteration:** For i = 1, 2, 3, ..., repeat:

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