MTH 617 Algebraic Number Theory

End Semester Examination

Total marks: 45 | December 3, 2002 | Time: $3 + \epsilon$ hours

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[*Each question carries equal points*]

1. Let $K \neq \mathbb{Q}$ be a real number field. Show that for any real $\epsilon > 0$ there exists $\alpha \in \mathfrak{O}_K$ such that $0 < \alpha < \epsilon$.

2. Let α be an algebraic integer, $f(X) := \operatorname{minpoly}_{\alpha,\mathbb{Q}}(X) \in \mathbb{Z}[X]$, $d := \deg f \geqslant 2$, $K := \mathbb{Q}(\alpha)$, $a, b \in \mathbb{Z}$ and $b \neq 0$. Show that $\operatorname{Tr}_{K|\mathbb{Q}}(a + b\alpha) = ad + b \operatorname{Tr}_{K|\mathbb{Q}}(\alpha)$ and that $\mathrm{N}_{K|\mathbb{Q}}(a + b\alpha) = (-b)^d f(-a/b)$.

3. Let K be a number field of degree 3. Show that $\Delta_K < -12$ or $\Delta_K > 20$.

4. Let A be an integral domain. Show that A is a Dedekind domain, if and only if A satisfies all of the following three conditions:

(1) A is Noetherian.

(2) A is integrally closed.

(3) Every descending chain of ideals of $A/\mathfrak{a}$ is stationary for every non-zero ideal $\mathfrak{a}$ of A.

(**Remark:** A descending chain of ideals $\mathfrak{a}_n$ of a ring R is a chain of the form $\mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \cdots \supseteq \mathfrak{a}_n \supseteq \cdots$. Such a chain is called stationary, if for some $n \in \mathbb{N}$ we have $\mathfrak{a}_{i+1} = \mathfrak{a}_i$ for all $i \geqslant n$. A ring in which every descending chain of ideals is stationary is commonly called an Artinian ring.)

5. Let p be a rational prime, ω a primitive p-th root of unity and $K := \mathbb{Q}(\omega)$. Show that the ideal $\langle p \rangle = p\mathfrak{O}_K$ factorizes in $\mathfrak{O}_K$ as $\mathfrak{q}^{p-1}$, where $\mathfrak{q} := \langle \omega - 1 \rangle = (\omega - 1)\mathfrak{O}_K$.

6. Let K be a number field and $\mathfrak{a}$ an ideal of $\mathfrak{O} := \mathfrak{O}_K$. Assume that there exists $\alpha \in \mathfrak{O}$ with $\mathrm{N}(\mathfrak{a}) = |\mathrm{N}(\alpha)|$. Prove or disprove: $\mathfrak{a} = \mathfrak{O}\alpha$.

7. Let $\mathfrak{a}$ be a non-zero ideal in $\mathfrak{O} := \mathfrak{O}_K$ for a number field K. Show that every ideal class of K contains an integral ideal $\mathfrak{b}$ with $\mathfrak{a} + \mathfrak{b} = \mathfrak{O}$.

8. Let $m > 1$ and $n > 1$ be two relatively prime rational integers, ω_m a primitive m-th root of unity and ω_n a primitive n-th root of unity. Show that $1 - \omega_m\omega_n$ is a unit in $\mathfrak{O}_{\mathbb{Q}(\omega_m, \omega_n)}$.

9. Let $D > 1$ be a square-free integer congruent to 1 or 2 modulo 4 and *not* of the form $3m^2 \pm 1$ for some integer m. Also assume that 3 does not divide the class number of $\mathbb{Q}(\sqrt{-D})$. Show that the (generalized) Bachet equation $y^2 = x^3 - D$ does not have solutions in (rational) integers.

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